Correlation functions of boundary operators:

the boundary form factor program

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Form factor bootstrap

Correlation functions in the bulk

$$\begin{aligned}\langle 0|\mathcal{O}_1(x)\mathcal{O}_2(0)|0\rangle &= \sum_{\psi}\langle 0|\mathcal{O}_1(x)|\psi\rangle\langle\psi|\mathcal{O}_2(0)|0\rangle \\ &= \sum_{n=0}^{\infty}\int\frac{d\vartheta_1}{2\pi}\ldots\int\frac{d\vartheta_n}{2\pi}\langle 0|\mathcal{O}_1(0)|\vartheta_1\ldots\vartheta_n\rangle\times \\ &\quad \langle\vartheta_1\ldots\vartheta_n|\mathcal{O}_2(0)|0\rangle\exp\left\{-mr\sum_{i=1}^{n}\cosh\vartheta_j\right\}\end{aligned}$$

Form factors

$$F_N^{\mathcal{O}}(\vartheta_1\ldots\vartheta_n) = \langle 0|\mathcal{O}(0)|\vartheta_1\ldots\vartheta_n\rangle$$

Integrable models: $F_n^{\mathcal{O}}$ are analytic functions satisfying certain axioms $\longrightarrow$ form factor bootstrap.

Boundary QFTs

$$\mathcal{A} = \int_{-\infty}^{\infty} dt \int_{-\infty}^{0} dx \left\{ \frac{1}{2} \partial_\mu \Phi \partial^\mu \Phi - V(\Phi) \right\} - \int_{-\infty}^{\infty} dt \, U(\Phi(t, x = 0))$$

1. Bulk operators: form factors complicated, position dependent e.g. $\langle \Phi(x) \rangle$

2. Boundary operators: much simpler, but no analogue of form factor bootstrap so far.

Necessary ingredients

1. Construction of asymptotic states

2. Boundary (LSZ) reduction formulae

See in: Z. Bajnok, G. Böhm and G. Takács
J. Phys. **A35** (2002) 9333-9342, hep-th/0207079
Nucl. Phys. **B682** (2004) 585-617, hep-th/0309119

$$
\begin{aligned}
{}_{out}\langle A|\,\mathcal{O}|k;B\rangle_{in} &= {}_{out}\langle A|\,\mathcal{O}A^{in}(k)^{\dagger}|B\rangle_{in} = \\
&= \text{disconnected part} + 2iZ^{-1/2}\int_{-\infty}^{0} dx\,\cos(kx)\mathrm{e}^{-i\omega(k)t}\times \\
&\quad \left\{\partial_t^2-\partial_x^2+m^2+\delta(x)\partial_x\right\}\,{}_{out}\langle A|\,T\mathcal{O}\Phi_\alpha(t,x)|B\rangle_{in}
\end{aligned}
$$

Boundary form factors

$$_{out}\langle\theta_1',\ldots,\theta_m'|\mathcal{O}(t)|\theta_1,\ldots,\theta_n\rangle_{in} = F^{\mathcal{O}}_{mn}(\theta_1',,\ldots,\theta_m';\theta_1,\ldots,\theta_n)e^{-imt(\sum\cosh\theta_i-\sum\cosh\theta_j')}$$

defined only for $\theta_1 > \cdots > \theta_n > 0$ and $\theta_1' < \cdots < \theta_m' < 0$, but can be continued as

$$_{out}\langle\theta_1',\ldots,\theta_m'|\mathcal{O}(t)|-\theta_1,\ldots,-\theta_n\rangle_{out} = F^{\mathcal{O}}_{mn}(\theta_1',\ldots,\theta_m';-\theta_1,\ldots,-\theta_n)$$

Crossing property: the "master" relation

$$F^{\mathcal{O}}_{mn}(\theta_1',\theta_2',\ldots,\theta_m';\theta_1,\ldots,\theta_n) = F^{\mathcal{O}}_{m-1n+1}(\theta_2',\ldots,\theta_m';\theta_1'+i\pi,\theta_1,\ldots,\theta_n)+\text{disc}$$

All can be expressed in terms of

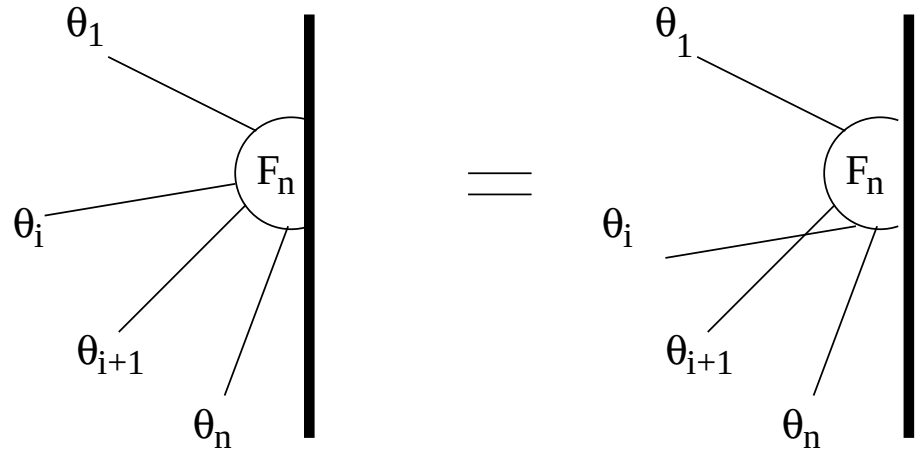
$$_{out}\langle 0|\mathcal{O}(0)|\theta_1,\theta_2,\ldots,\theta_n\rangle_{in} = F^{\mathcal{O}}_n(\theta_1,\theta_2,\ldots,\theta_n)$$

Boundary form factor axioms

BFF axioms: follow by continuing to complex rapidities using bLSZ.

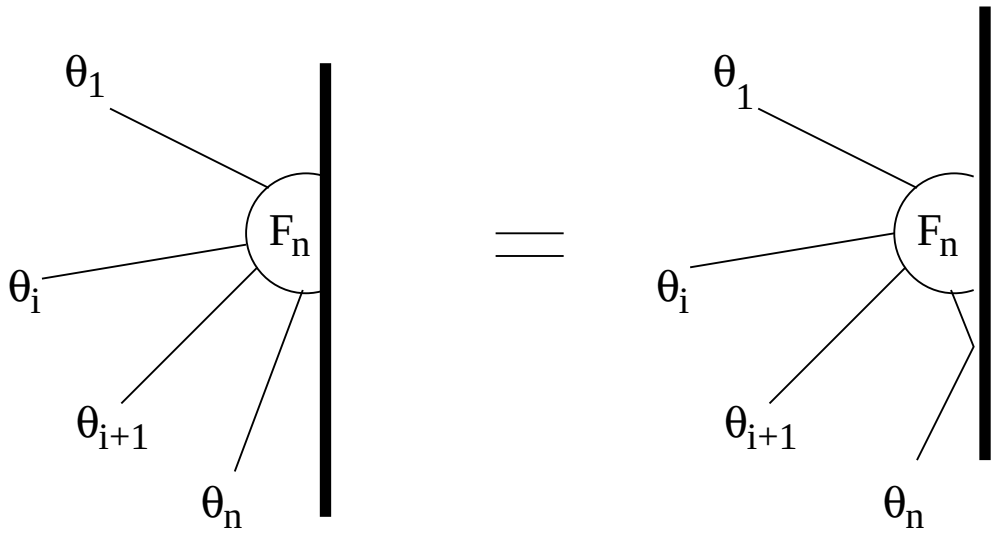
I. Permutation:

$$F_n^{\mathcal{O}}(\theta_1,\ldots,\theta_i,\theta_{i+1},\ldots,\theta_n) = S(\theta_i-\theta_{i+1})F_n^{\mathcal{O}}(\theta_1,\ldots,\theta_{i+1},\theta_i,\ldots,\theta_n)$$



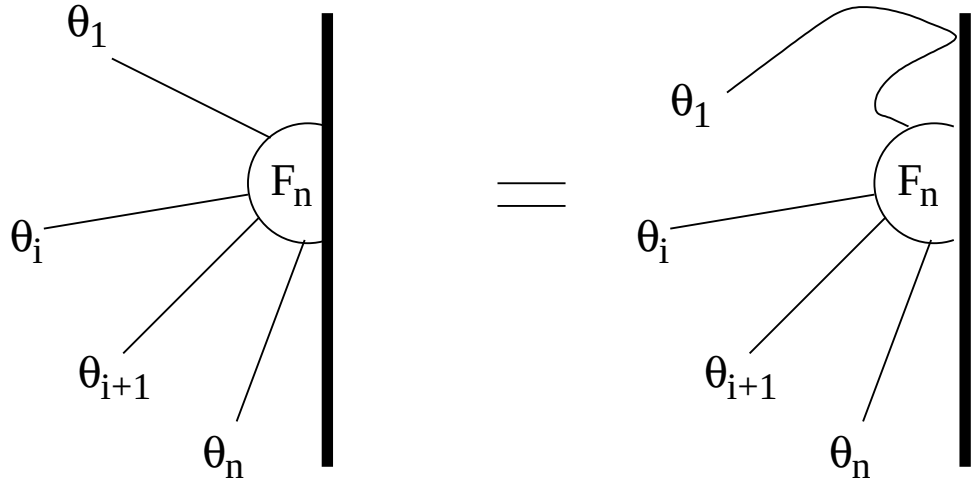
II. Reflection:

$$F_n^{\mathcal{O}}(\theta_1, \ldots, \theta_{n-1}, \theta_n) = R(\theta_n) F_n^{\mathcal{O}}(\theta_1, \ldots, \theta_{n-1}, -\theta_n)$$



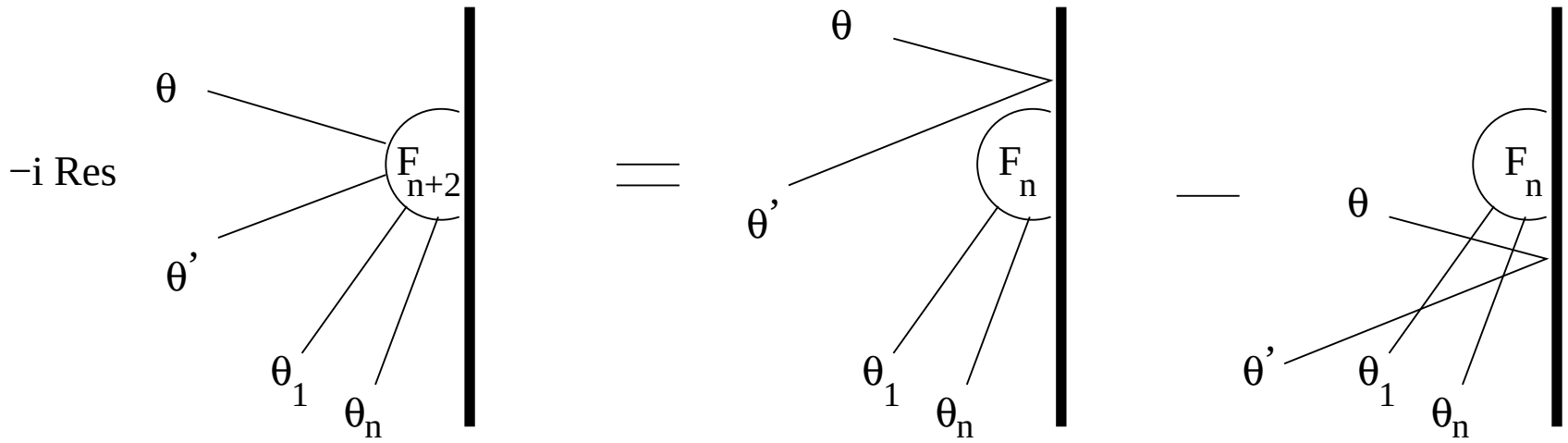
III. Crossing reflection:

$$F_n^{\mathcal{O}}(\theta_1, \theta_2, \ldots, \theta_n) = R(i\pi - \theta_1) F_n^{\mathcal{O}}(2i\pi - \theta_1, \theta_2, \ldots, \theta_n)$$



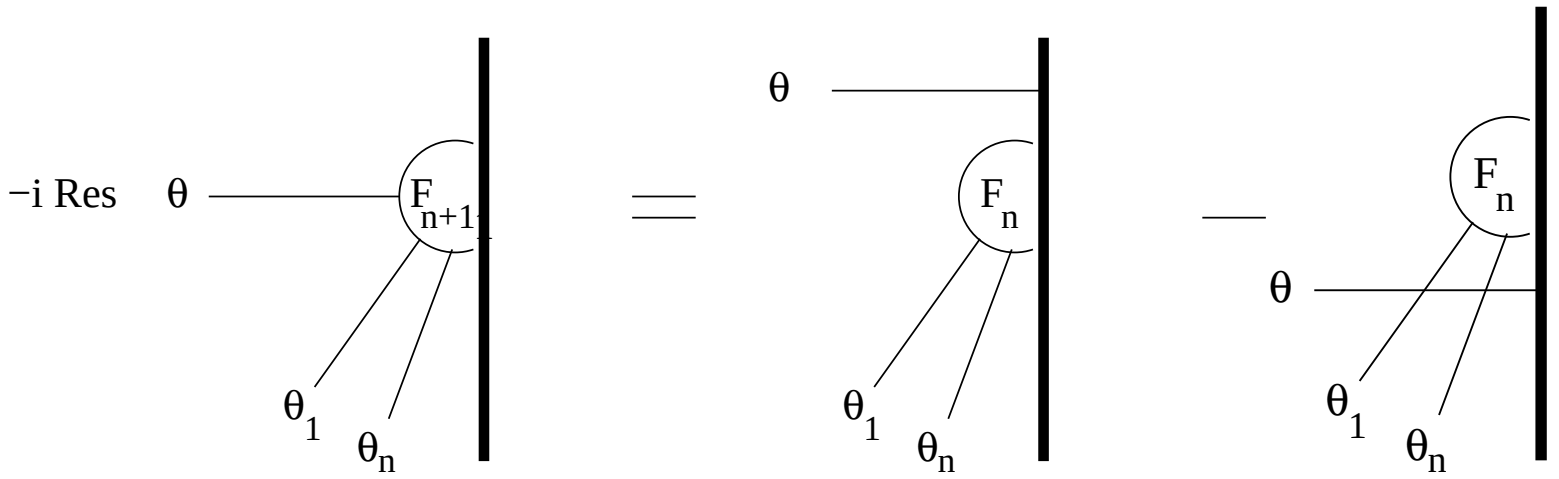
IV. Kinematical singularity

$$-i \operatorname*{Res}_{\theta=\theta'} F^{\mathcal{O}}_{n+2}(-\theta+i\pi,\theta',\theta_1,\ldots,\theta_n) = \left(R(\theta)-\prod_{i=1}^{n} S(\theta-\theta_i)R(\theta)S(\theta+\theta_i)\right)\times F^{\mathcal{O}}_n(\theta_1,\ldots,\theta_n)$$



V. Boundary kinematical singularity: $R(\theta) \sim \frac{ig^2/2}{\theta - i\pi/2}$

$$-i \operatorname*{Res}_{\theta=0} F^{\mathcal{O}}_{n+1}(\theta + \frac{i\pi}{2}, \theta_1, \ldots, \theta_n) = \frac{g}{2}\Big(1 - \prod_{i=1}^{n} S\big(\frac{i\pi}{2} - \theta_i\big)\Big) F^{\mathcal{O}}_n(\theta_1, \ldots, \theta_n)$$

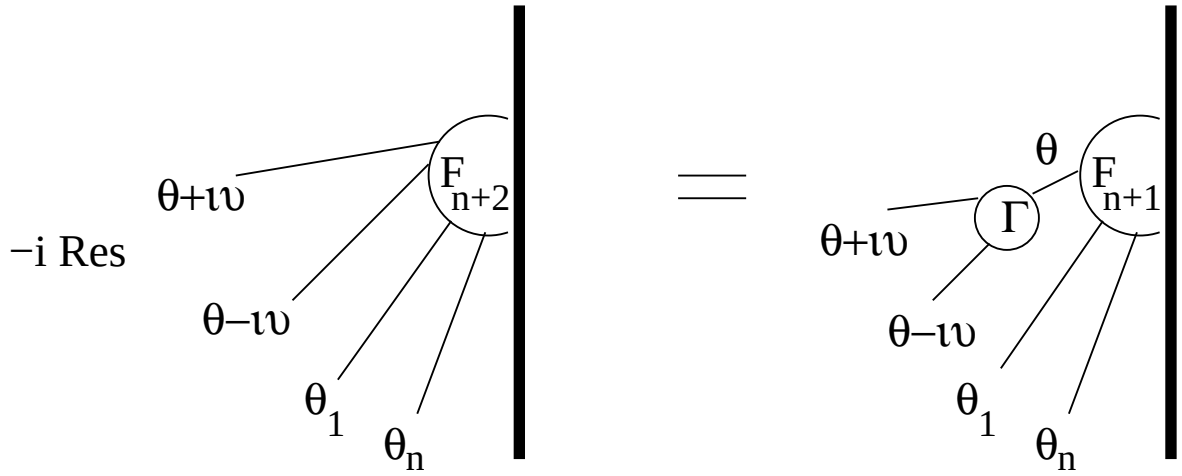


sine-Gordon: free-field representation in B. Hou et al. *Int. J. Mod. Phys.* **A12** (1997) 1711-1741 entirely misses this axiom!

VI. Bulk dynamical singularity:

$$S(\theta) \sim \frac{i\Gamma^2}{\theta - iu}$$

$$-i \operatorname*{Res}_{\theta=\theta'} F^{\mathcal{O}}_{n+2}(\theta + iu, \theta' - iu, \theta_1, \ldots, \theta_n) = \Gamma F^{\mathcal{O}}_{n+1}(\theta, \theta_1, \ldots, \theta_n)$$



VII. Boundary dynamical singularity:

$$R(\theta) \sim \frac{i\tilde{g}^2/2}{\theta - iv}$$

$$-i \operatorname*{Res}_{\theta=iv} F^{\mathcal{O}}_{n+1}(\theta_1, \ldots, \theta_n, \theta) = \tilde{g}\tilde{F}^{\mathcal{O}}(\theta_1, \ldots, \theta_n)$$

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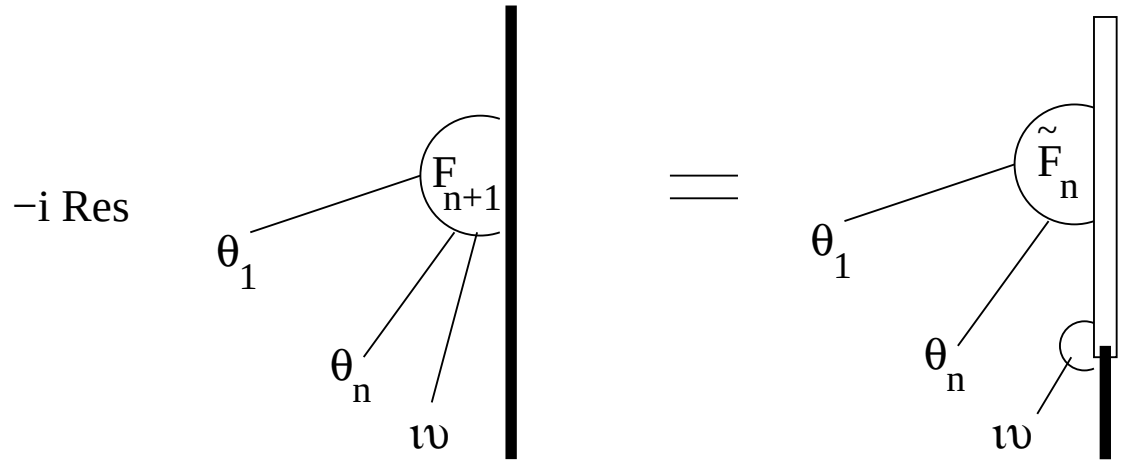


Consistency check

From the bulk kinematical singularity equation

$$i \operatorname*{Res}_{\theta=\frac{i\pi}{2}} i \operatorname*{Res}_{\theta'=\theta} F^{\mathcal{O}}_{n+2}(-\theta+i\pi,\theta',\theta_1,\ldots,\theta_n) =$$

$$\left(-i \operatorname*{Res}_{\theta=\frac{i\pi}{2}} R(\theta)\right)\left(1-\prod_{i=1}^{n} S(\frac{i\pi}{2}-\theta_i)S(\frac{i\pi}{2}+\theta_i)\right) F^{\mathcal{O}}_{n}(\theta_1,\ldots,\theta_n)$$

Using boundary kinematical singularity equation twice

$$i \operatorname*{Res}_{\theta=\frac{i\pi}{2}} i \operatorname*{Res}_{\theta'=\frac{i\pi}{2}} F^{\mathcal{O}}_{n+2}(-\theta+i\pi,\theta',\theta_1,\ldots,\theta_n) =$$

$$\frac{g^2}{4}\left(1+\prod_{i=1}^{n} S(\frac{i\pi}{2}-\theta_i)\right)\left(1-\prod_{i=1}^{n} S(\frac{i\pi}{2}-\theta_i)\right) F^{\mathcal{O}}_{n}(\theta_1,\ldots,\theta_n)$$

Factor 2 : $F^{\mathcal{O}}_{n+2}(\theta,\theta,\theta_1,\ldots,\theta_n)=0 \quad \Leftarrow \quad S(0)=-1$

General solution of the axioms

One-particle minimal form factor

$$r(\theta) = R(\theta) r(-\theta) \quad ; \quad r(i\pi + \theta) = R(-\theta) r(i\pi - \theta)$$

Can be solved following Karowski & Weisz: minimal solution
Bulk minimal two-particle form factor

$$f(\theta) = S(\theta) f(-\theta), \qquad f(i\pi + \theta) = f(i\pi - \theta)$$

General ansatz:

$$F_n(\theta_1, \theta_2, \ldots, \theta_n) = \frac{Q_n(y_1, y_2 \ldots, y_n)}{\prod\limits_{i<j} (y_i + y_j)} \prod_{i=1}^{n} r(\theta_i) \prod_{i<j} f(\theta_i - \theta_j) f(\theta_i + \theta_j)$$

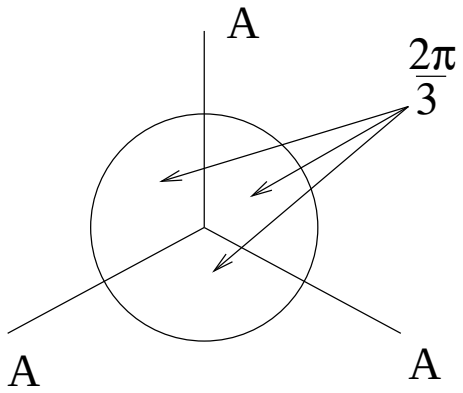
Q_n: symmetric function of variables $y_i = 2 \cosh \theta_i$

Singularity axioms $\Rightarrow$ recursion relations for Q_n

1. Bulk kinematical singularity axiom relates Q_n to Q_{n-2}

2. Boundary kinematical singularity axiom relates Q_n to Q_{n-1}

3. Bulk self fusion pole ($u = \frac{2\pi}{3}$) relates Q_n to Q_{n-1}



Asymptotic behaviour

Form factor expansion of correlation functions

$$\langle 0|\mathcal{O}(t)\mathcal{O}(0)|0\rangle = \sum_{n=0}^{\infty} \frac{1}{(2\pi)^n} \int_{\theta_1>\theta_2>\cdots>\theta_n>0} d\theta_1 d\theta_2 \dots d\theta_n e^{-imt\sum_i \cosh\theta_i} F_n F_n^+$$

$$F_n = \langle 0|\mathcal{O}(0)|\theta_1,\theta_2,\dots,\theta_n\rangle_{in} = F_n^{\mathcal{O}}(\theta_1,\theta_2,\dots,\theta_n)$$

$$F_n^+ = {}_{in}\langle\theta_1,\theta_2,\dots,\theta_n|\mathcal{O}(0)|0\rangle = F_n^{\mathcal{O}}(i\pi+\theta_n, i\pi+\theta_{n-1},\dots,i\pi+\theta_1)$$

$(= F_n^*$ for unitary theories)

Power-like short distance asymptotics

$$\langle 0|\mathcal{O}(t)\mathcal{O}(0)|0\rangle \sim t^{-2\Delta} \qquad t\to 0$$

$$\Downarrow$$

$$F_n^{\mathcal{O}}(\theta_1,\theta_2,\dots,\theta_n) \sim \mathrm{e}^{N\theta_i} \qquad \theta_i\to+\infty$$

naively : $\Delta = N$

Free boson with Robin BC

Action

$$\mathcal{A} = \int_{-\infty}^{\infty} dt \int_{-\infty}^{0} dx \left(\frac{1}{2}(\partial_t \Phi)^2 - \frac{1}{2}(\partial_x \Phi)^2 - \frac{m^2}{2}\Phi^2 \right) - \int_{-\infty}^{\infty} dt \frac{\lambda}{2}(\Phi - \Phi_0)^2$$

$$S = 1 \qquad , \qquad R(\theta) = \frac{\sinh\theta - i\lambda/m}{\sinh\theta + i\lambda/m}$$

Exact solution of QFT: form factor of generic boundary operator

$$\langle 0| : \partial_{\tau_1}^{n_1}\Phi(0,0) \ldots \partial_{\tau_k}^{n_k}\Phi(0,0) : |\theta_1 \ldots \theta_k\rangle =$$
$$(m\cosh\theta_1)^{n_1}(1+R(\theta_1)) \ldots (m\cosh\theta_k)^{n_k}(1+R(\theta_k)) + \text{permutations}$$

BFF boostrap: $r(\theta) = 1 + R(\theta)$

Polynomial part: Q_n is any symmetric polynomial of $y_i = 2\cosh\theta_i \Rightarrow$ the same space of form factors.

Free fermion: Ising with boundary magnetic field

Scattering amplitudes

$$S(\theta) = -1 \qquad , \qquad R(\theta) = i \tanh\left(\frac{i\pi}{4} - \frac{\theta}{2}\right) \frac{\sinh\theta + i\kappa}{\sinh\theta - i\kappa} \qquad , \quad \kappa = 1 - \frac{h^2}{2m}$$

Exact solution for one-particle form factor

$$\langle 0|\psi_\pm(0,t)|\theta\rangle = e^{-i\omega(\theta)t}\left(u_\pm(\theta) + R(\theta)u_\pm(-\theta)\right) \quad , \quad u_\pm(\theta) = \sqrt{m}e^{\mp\frac{i\pi+2\theta}{4}}$$

Operator space: $\partial_\tau^{n_1}\psi_+\partial_\tau^{n_2}\psi_+\ldots\partial_\tau^{n_k}\psi_+$ with $n_1 > n_2 > \cdots > n_k$
and UV dimension

$$\Delta = k/2 + \sum_{i=1}^{k} n_i$$

BFF minimal form factor: $r(\theta) = \frac{\sinh\theta}{\sinh(\frac{\theta}{2}+i\frac{\pi}{4})}\frac{\cosh\theta+i(1-\kappa)}{\sinh\theta-i\kappa}$ $\surd$

Dimension of solution space with given Δ: $\surd$

Boundary Lee-Yang model

Action

$$\mathcal{A}_{\lambda,\Phi(h)} = \mathcal{A}_\Phi + \lambda \int_{-\infty}^{\infty} dt \int_{-\infty}^{0} dx \phi(x,y) + h \int_{-\infty}^{\infty} dt \varphi(y)$$

$\mathcal{A}_\Phi$: $\mathcal{M}(2/5)$ with the Φ boundary condition

$$S(\theta) = \frac{\sinh\theta + i\sin 2\pi/3}{\sinh\theta - i\sin 2\pi/3}$$

Minimal bulk two particle form factor:

$$f(\theta) = \frac{y-2}{y+1} v(i\pi-\theta)v(-i\pi+\theta)\,,\; v(\theta) = \exp\left\{2\int_0^\infty \frac{dt}{t} e^{i\frac{\theta t}{\pi}} \frac{\sinh\frac{t}{2}\sinh\frac{t}{3}\sinh\frac{t}{6}}{\sinh^2 t}\right\}$$

Minimal boundary form factor

Reflection factor

$$R(\theta)_\Phi = R_0(\theta)R(b,\theta) = \left(\frac{1}{2}\right)\left(\frac{1}{6}\right)\left(-\frac{2}{3}\right)\left[\frac{b+1}{6}\right]\left[\frac{b-1}{6}\right]$$

$$h(b) = \sin\left(\left(b+\frac{1}{2}\right)\frac{\pi}{5}\right) m(\lambda)^{6/5} h_{crit}, \quad h_{crit} = -\frac{\pi^{\frac{3}{5}} 2^{\frac{4}{5}} 5^{\frac{1}{4}} \sin\frac{2\pi}{5}}{\sqrt{\Gamma(\frac{3}{5})\Gamma(\frac{4}{5})}} \left(\frac{\Gamma(\frac{3}{5})}{\Gamma(\frac{3}{5})}\right)^{\frac{6}{5}}$$

1PFF:

$$r(\theta) = \frac{i\sinh\theta}{(\sinh\theta - i\sin\frac{\pi(b+1)}{6})(\sinh\theta - i\sin\frac{\pi(b-1)}{6})} u(\theta)$$

$$u(\theta) = \exp\left\{\int_0^\infty \frac{dt}{t}\left[\frac{1}{\sinh\frac{t}{2}} - 2\cosh\frac{t}{2}\cos\left[\left(\frac{i\pi}{2}-\theta\right)\frac{t}{\pi}\right]\frac{\sinh\frac{5t}{6}+\sinh\frac{t}{2}-\sinh\frac{t}{3}}{\sinh^2 t}\right]\right.$$

Solution with minimal Δ

General n-particle form factors:

$$F_n(\theta_1,\ldots,\theta_n) \;=\; H_n Q_n(y_1,\ldots,y_n)\prod_i \frac{r(\theta_i)}{y_i}\prod_{i<j}\frac{f(\theta_i-\theta_j)f(\theta_i+\theta_j)}{y_i+y_j}$$

$$H_n \;=\; N\left(\frac{i3^{\frac{1}{4}}}{2^{\frac{1}{2}}v(0)}\right)^n \qquad \beta_k(b)=2\cos\frac{\pi}{6}(b+k)$$

Minimal solution of recursion equations up to $n=3$: *unique*

$$Q_1(y_1)=\sigma_1, \qquad Q_2(y_1,y_2)=\sigma_1(\sigma_2+3-\beta_{-3}^2),$$
$$Q_3(y_1,y_2,y_3)=\sigma_1\left[\sigma_1(\sigma_2+\beta_{-1}^2)(\sigma_2+\beta_1^2)-(\sigma_2+3)(\sigma_1\sigma_2-\sigma_3)\right].$$

$F_n^{\mathcal{O}}(\theta_1,\ldots,\theta_n)\sim O(1)$ as $\theta_i\to\infty$: naively $\Delta_{\mathcal{O}}=0$!

Comparing with QFT

Operator of minimal dimension: $\phi(t)$, $\Delta = -1/5$

$$\langle 0|\mathcal{O}(t)\mathcal{O}(0)|0\rangle = \sum_{n=0}^{\infty} \frac{1}{(2\pi)^n} \int_{\theta_1>\theta_2>\cdots>\theta_n>0} d\theta_1 d\theta_2 \dots d\theta_n e^{-imt\sum_i \cosh\theta_i} F_n^{\mathcal{O}} F_n^{\mathcal{O}+}$$

develops anomalous dimension under resummation due to logarithmic corrections:

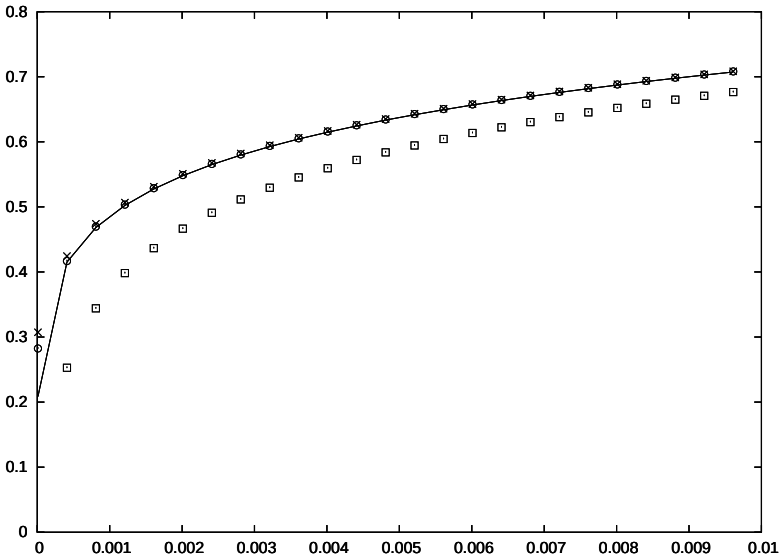
$$t^N(1 + \alpha \log t + \frac{1}{2}\alpha^2 \log^2 t + \dots) = t^{N+\alpha}$$

QFT prediction

$$\langle 0|m^{\frac{1}{5}}\varphi(t)m^{\frac{1}{5}}\varphi(0)|0\rangle = -(mt)^{\frac{2}{5}} + (mt)^{\frac{1}{5}} C^{\varphi}_{\varphi\varphi}\langle m^{\frac{1}{5}}\varphi\rangle + \dots$$

$$C^{\varphi}_{\varphi\varphi} = -\sqrt{\frac{1+\sqrt{5}}{2}}\sqrt{\frac{\Gamma(\frac{1}{5})\Gamma(\frac{6}{5})}{\Gamma(\frac{3}{5})\Gamma(\frac{4}{5})}} \quad , \quad \langle m^{\frac{1}{5}}\varphi\rangle = -\frac{5}{6h_{crit}}\frac{\cos(\frac{\pi b}{6})}{\cos(\frac{\pi}{10}(2b+1))}$$

Numerical comparison



$b = -1.05$, dimensionless correlation plotted against mt

$$\Downarrow$$

$$\mathcal{O}(t) = m^{1/5}\phi(t) \quad \surd$$

Other models

1. Lee-Yang model with BC $\mathbf{1}$
   Lowest nontrivial operator: $\theta(t) \quad (= L_{-2}\mathrm{Id})$ $\surd$

2. sinh-Gordon model with Dirichlet BC
   free field limit $\surd$
   qualitative structure of simplest solutions $\surd$

Summary and outlook

1. Boundary form factor bootstrap initiated
   Framework set up: complete set of form factor axioms known

2. Validity of axioms demonstrated
   Free field theories: free boson (Robin), free fermion (Ising)
   Interacting theories: scaling Lee-Yang, sinh-Gordon

3. Extension of axioms to nondiagonal scattering is straightforward

Open directions

1. Obtaining complete solution:
   operator counting for Lee-Yang/sinh-Gordon (in progress)

2. Further checks

3. Sum rules (g/Δ-theorem$\sim$ bulk c/Δ-theorem)

4. Extension to massless case: description boundary RG flows

5. Finding applications e.g. to condensed matter physics