THE SPECTRUM OF BOUNDARY SINE-GORDON THEORY *

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Abstract We review our recent results on the on-shell description of sine-Gordon model with integrable boundary conditions. We determined the spectrum of boundary states by closing the boundary bootstrap and gave a derivation of Al.B. Zamolodchikov's (unpublished) formulae for the boundary energy and the relation between the Lagrangian (ultraviolet) and bootstrap (infrared) parameters. These results have been checked against numerical finite volume spectra coming from the truncated conformal space approach. We find an entirely consistent picture and strong evidence for the validity of the conjectured spectrum and scattering amplitudes, which together give a complete description of the boundary sine-Gordon theory on mass shell.

Keywords: Integrable field theory, field theory with boundary, bootstrap, perturbed conformal field theory, sine-Gordon model

1. Introduction

In these proceedings we report on the results of our works [1, 2, 3]. Instead of following the line of the original conference talk, we wish to give a summary of the results and conjectures which together describe the on-shell spectral data of boundary sine-Gordon theory.

Sine-Gordon field theory is one of the most important quantum field theoretic models with numerous applications ranging from particle theoretic problems to condensed matter systems, and one which has played a central role in our understanding of $1+1$ dimensional field theories. A crucial property of the model is integrability, which permits an exact analytic determination of many of its physical properties and characteristic quantities. Integrability can also be preserved in the presence of bound-

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aries [4]; for sine-Gordon theory, the most general boundary potential that preserves integrability was found by Ghoshal and Zamolodchikov [5]. They also introduced the notion of 'boundary crossing unitarity', and combining it with the boundary version of the Yang-Baxter equations they were able to determine soliton reflection factors on the ground state boundary; later Ghoshal completed this work by determining the breather reflection factors [6].

The first (partial) results on the spectrum of the excited boundary states were obtained by Saleur and Skorik for Dirichlet boundary conditions [7]. However, they did not take into account the boundary analogue of the Coleman-Thun mechanism, the importance of which was first emphasized by Dorey et al. [8]. Using this mechanism Mattsson and Dorey were able to close the bootstrap in the Dirichlet case and determine the complete spectrum and the reflection factors on the excited boundary states [9]. Recently we used their ideas to obtain the spectrum of excited boundary states and their reflection factors for the Neumann boundary condition [1] and then for the general two-parameter family of integrable boundary conditions [2].

Another interesting problem is the relation between the ultraviolet (UV) parameters that appear in the perturbed CFT Hamiltonian and the infrared (IR) parameters in the reflection factors. This relation was first obtained by Al. B. Zamolodchikov [10] together with a formula for the boundary energy; however, his results remain unpublished. In order to have these formulae, we rederived them in our paper [3], where we used them to check the consistency of the spectrum and of the reflection factors against a boundary version of truncated conformal space approach (TCSA). Combining the TCSA results with analytic methods of the Bethe Ansatz we found strong evidence that our understanding of the spectrum of boundary sine-Gordon model is indeed correct.

2. Boundary bootstrap in sine-Gordon theory

Boundary sine-Gordon theory is defined by the action

$$\begin{aligned}\mathcal{A}_{sG} &= \int_{-\infty}^{\infty} dt \Bigg(\int_{-\infty}^{0} dx \left[\frac{1}{2} \partial_\mu \Phi \partial^\mu \Phi + \frac{m_0^2}{\beta^2} \cos \beta \Phi \right] \\ &+ M_0 \cos \frac{\beta}{2} \left(\Phi(0,\, t) - \phi_0 \right) \Bigg) \end{aligned} \tag{1}$$

where $\Phi(x,\, t)$ is a real scalar field and $M_0,\ \phi_0$ are the two parameters characterizing the boundary condition:

$$\partial_x \Phi(x,t)|_{x=0} = -M_0 \frac{\beta}{2} \sin \left(\frac{\beta}{2} (\Phi(0,t) - \phi_0) \right) . \tag{2}$$

Ghoshal and Zamolodchikov showed that the above model is integrable [5] and that the boundary term is the most general consistent with integrability and containing no time derivatives of the field Φ.

In the bulk sine-Gordon model the particle spectrum consists of the soliton s, the antisoliton $\bar{s}$, and the breathers B^n, which appear as bound states of a soliton and an antisoliton. As a consequence of the integrable nature of the model any scattering amplitude factorizes into a product of two particle scattering amplitudes which were found by Zamolodchikov and Zamolodchikov [11]. Factorization of the scattering carries over to the situation with integrable boundary conditions as well [5].

2.1 Ground state reflection factors

In the presence of boundary, the bulk S-matrix must be supplemented with the reflection amplitudes describing the interaction of the particles with the boundary in order to specify the scattering theory completely. The most general reflection factor - modulo CDD-type factors - of the soliton antisoliton multiplet $|s,\bar{s}\rangle$ on the ground state boundary, denoted by $|\,\rangle$, satisfying the boundary versions of the Yang-Baxter, unitarity and crossing equations was found by Ghoshal and Zamolodchikov [5]:

$$\begin{aligned}
R(\eta,\vartheta,u) &= \begin{pmatrix} P^+(\eta,\vartheta,u) & Q(\eta,\vartheta,u) \\ Q(\eta,\vartheta,u) & P^-(\eta,\vartheta,u) \end{pmatrix} \\
&= \begin{pmatrix} P_0^+(\eta,\vartheta,u) & Q_0(u) \\ Q_0(u) & P_0^-(\eta,\vartheta,u) \end{pmatrix} R_0(u)\frac{\sigma(\eta,u)}{\cos(\eta)}\frac{\sigma(i\vartheta,u)}{\cosh(\vartheta)}\ , \\
P_0^\pm(\eta,\vartheta,u) &= \cos(\lambda u)\cos(\eta)\cosh(\vartheta) \mp \sin(\lambda u)\sin(\eta)\sinh(\vartheta) \\
Q_0(u) &= -\sin(\lambda u)\cos(\lambda u)
\end{aligned} \tag{3}$$

where we introduced

$$\lambda = \frac{8\pi}{\beta^2} - 1\,, \tag{4}$$

$u = -i\theta$ denote the purely imaginary rapidity as in [5], η and ϑ are two real parameters characterizing the solution,

$$R_0(u) = \prod_{l=1}^{\infty}\left[\frac{\Gamma(4l\lambda - \frac{2\lambda u}{\pi})\Gamma(4\lambda(l-1)+1-\frac{2\lambda u}{\pi})}{\Gamma((4l-3)\lambda - \frac{2\lambda u}{\pi})\Gamma((4l-1)\lambda+1-\frac{2\lambda u}{\pi})}/(u\to -u)\right]$$

is the boundary condition independent part and

$$\sigma(x,u) = \frac{\cos x}{\cos(x+\lambda u)}$$
$$\prod_{l=1}^{\infty}\left[\frac{\Gamma(\frac{1}{2}+\frac{x}{\pi}+(2l-1)\lambda-\frac{\lambda u}{\pi})\Gamma(\frac{1}{2}-\frac{x}{\pi}+(2l-1)\lambda-\frac{\lambda u}{\pi})}{\Gamma(\frac{1}{2}-\frac{x}{\pi}+(2l-2)\lambda-\frac{\lambda u}{\pi})\Gamma(\frac{1}{2}+\frac{x}{\pi}+2l\lambda-\frac{\lambda u}{\pi})}/(u\to -u)\right]$$

describes the boundary condition dependence. Minimality (i.e. minimal pole structure) restricts $0 \leq \eta \leq \pi(\lambda+1)/2$, while the independent values of ϑ are $0 \leq \vartheta \leq \infty$. As it can be seen from the UV-IR relation to be discussed later (eqn. (11)), this covers exactly the range of parameters in the Lagrangian description; therefore it is thought that only the minimal solution is realized in boundary sine-Gordon model. This is also confirmed by our TCSA analysis (see Section 4).

As a consequence of the bootstrap equations [5] the breather reflection factors share the structure of the solitonic ones, [6]:

$$R^{(n)}(\eta,\vartheta,u) = R_0^{(n)}(u) S^{(n)}(\eta,u) S^{(n)}(i\vartheta,u) \ , \tag{5}$$

where

$$\begin{aligned} R_0^{(n)}(u) &= \frac{\left(\frac{1}{2}\right)\left(\frac{n}{2\lambda}+1\right)}{\left(\frac{n}{2\lambda}+\frac{3}{2}\right)} \prod_{l=1}^{n-1} \frac{\left(\frac{l}{2\lambda}\right)\left(\frac{l}{2\lambda}+1\right)}{\left(\frac{l}{2\lambda}+\frac{3}{2}\right)^2} \\ S^{(n)}(x,u) &= \prod_{l=0}^{n-1} \frac{\left(\frac{x}{\lambda\pi}-\frac{1}{2}+\frac{n-2l-1}{2\lambda}\right)}{\left(\frac{x}{\lambda\pi}+\frac{1}{2}+\frac{n-2l-1}{2\lambda}\right)} , \quad (x) = \frac{\sin\left(\frac{u}{2}+\frac{x\pi}{2}\right)}{\sin\left(\frac{u}{2}-\frac{x\pi}{2}\right)} . \end{aligned} \tag{6}$$

In general $R_0^{(n)}$ describes the boundary independent properties and the other factors give the boundary dependent ones.

2.2 The spectrum of boundary bound states and the associated reflection factors

In the general case, the spectrum of boundary bound states was derived in [2]. It is a straightforward generalization of the spectrum in the Dirichlet limit previously obtained by Mattsson and Dorey [9]. The states can be labeled by a sequence of integers $|n_1, n_2, \ldots, n_k\rangle$. Such a state exists whenever the

$$\frac{\pi}{2} \geq \nu_{n_1} > w_{n_2} > \nu_{n_3} > w_{n_4} > \ldots \geq 0$$

condition holds, where

$$\nu_n = \frac{\eta}{\lambda} - \frac{(2n+1)\pi}{2\lambda} \quad \text{and} \quad w_n = \pi - \frac{\eta}{\lambda} - \frac{(2n-1)\pi}{2\lambda} \ ,$$

denote the location of certain poles in $\sigma(\eta,u)$. The mass of such a state (i.e. its energy above the ground state) is

$$m_{|n_1,n_2,\ldots,n_k\rangle} = M \sum_{i \text{ odd}} \cos(\nu_{n_i}) + M \sum_{i \text{ even}} \cos(w_{n_i}) \ . \tag{7}$$

The reflection factors of the various particles on these boundary states depend on whether k is even or odd. When k is even, we have

$$Q_{|n_1,n_2,\ldots,n_k\rangle}(\eta,\vartheta,u)=Q(\eta,\vartheta,u)\prod_{i\ \mathrm{odd}}a_{n_i}(\eta,u)\prod_{i\ \mathrm{even}}a_{n_i}(\bar{\eta},u)\ ,$$

and

$$P^{\pm}_{|n_1,n_2,\ldots,n_k\rangle}(\eta,\vartheta,u)=P^{\pm}(\eta,\vartheta,u)\prod_{i\ \mathrm{odd}}a_{n_i}(\eta,u)\prod_{i\ \mathrm{even}}a_{n_i}(\bar{\eta},u)\ ,$$

for the solitonic processes, where

$$a_n(\eta,u)=\prod_{l=1}^{n}\left\{2\left(\frac{\eta}{\pi}-l\right)\right\}\quad;\quad\bar{\eta}=\pi(\lambda+1)-\eta\ .$$

and

$$\{y\}=\frac{\left(\frac{y+1}{2\lambda}\right)\left(\frac{y-1}{2\lambda}\right)}{\left(\frac{y+1}{2\lambda}-1\right)\left(\frac{y-1}{2\lambda}+1\right)}$$

For the breather reflection factors the analogous formula is

$$R^{(n)}_{|n_1,n_2,\ldots,n_k\rangle}(\eta,\vartheta,u)=R^{(n)}(\eta,\vartheta,u)\prod_{i\ \mathrm{odd}}b^{n}_{n_i}(\eta,u)\prod_{i\ \mathrm{even}}b^{n}_{n_i}(\bar{\eta},u)\qquad(8)$$

where now

$$b^n_k(\eta,u)=\prod_{l=1}^{\min(n,k)}\left\{\frac{2\eta}{\pi}-\lambda+n-2l\right\}\left\{\frac{2\eta}{\pi}+\lambda-n-2(k+1-l)\right\}\ .\qquad(9)$$

In the case when k is odd, the same formulae apply if in the $P^{\pm}$, Q and $R^{(n)}$ ground state reflection factors the $\eta\leftrightarrow\bar{\eta}$ and $s\leftrightarrow\bar{s}$ changes are made.

2.3 Closure of the bootstrap

In our papers [1, 2] we presented an argument that the bootstrap closes for the above spectrum. The essential steps are:

1 We conjectured the minimal spectrum (i.e. the states that are necessary to include) by examining the reflection amplitudes of the solitons.

2 We proved that these states must be included in the spectrum, i.e. that the poles in the reflection factors corresponding to them cannot be explained by any boundary Coleman-Thun diagram.

3 For all other poles of the breather and soliton reflection factors we found an explanation in terms of one of the states listed above or at least one boundary Coleman-Thun type diagram which had the same order as the pole.

The only thing that remains is to check that the full residues of the poles can indeed be obtained as sums of contributions of all possible diagrams, using only the states in the minimal spectrum. We checked this in some of the simplest cases explicitly [1]. However, finding all the diagrams and computing all the residues is a horrendous task, which we have not completed. From TCSA we have overwhelming evidence that the spectrum and the reflection factors are correct and we briefly discuss this evidence in the sequel.

For the case of the Neumann boundary conditions [1] we noted that the conjectured spectrum implies that there are poles (in breather reflection factors) whose residue can only be explained by including contributions both from a boundary excited state and from a Coleman-Thun type diagram. In boundary Lee-Yang model, a very similar phenomenon was discussed by Dorey, Tateo and Watts [8]. There it was also related to the fact that the closure of the bootstrap was not unique. In the case of the sine-Gordon theory, however, the phenomenon that a pole can only be explained by a combination of some Coleman-Thun diagram together with some boundary excited state, happens only for some special values of the parameter η and so we do not think that it is an indication of any nonuniqueness in the bootstrap. Indeed, for generic values of the parameters the bootstrap closure does seem to be uniquely determined and therefore we think that even for the special values the correct closure of the bootstrap is the one above, since we expect that the spectrum depends smoothly on the parameters η and ϑ.

3. Zamolodchikov's formulae

Recently, Al. B. Zamolodchikov presented [10] a formula for the relation between the UV and the IR parameters in the boundary sine-Gordon model. We shall consider boundary sine-Gordon theory as a joint bulk and boundary perturbation of the $c = 1$ free boson with Neumann boundary conditions (perturbed conformal field theory, pCFT):

$$\begin{aligned}\mathcal{A}_{pCFT} &= \mathcal{A}^{N}_{c=1} + \mu \int_{-\infty}^{\infty} dt \int_{-\infty}^{0} dx \ :\cos\beta\Phi(x,\,t): \\ &+ \ \tilde{\mu} \int_{-\infty}^{\infty} dt \ :\cos\frac{\beta}{2}\left(\Phi(0,\,t)-\phi_0\right): \end{aligned} \tag{10}$$

where the colons denote the standard CFT normal ordering, which defines the normalization of the operators and of the coupling constants. The couplings have nontrivial dimensions: $[\mu] = [\mathrm{mass}]^{2-\beta^2/4\pi}$, $[\tilde{\mu}] = [\mathrm{mass}]^{1-\beta^2/8\pi}$.

The UV parameters associated to the boundary are $\tilde{\mu}$ and ϕ_0, while the IR parameters are η and ϑ appearing in (3). The other UV parameter μ is related to the soliton mass M the same way as for the bulk theory [13]. With the above conventions the UV-IR relation [1] is

$$\begin{aligned} \cos\left(\frac{\beta^2\eta}{8\pi}\right)\cosh\left(\frac{\beta^2\vartheta}{8\pi}\right) &= \frac{\tilde{\mu}}{\sqrt{2\mu}}\sqrt{\sin\left(\frac{\beta^2}{8}\right)}\cos\left(\frac{\beta\phi_0}{2}\right)\,, \\ \sin\left(\frac{\beta^2\eta}{8\pi}\right)\sinh\left(\frac{\beta^2\vartheta}{8\pi}\right) &= \frac{\tilde{\mu}}{\sqrt{2\mu}}\sqrt{\sin\left(\frac{\beta^2}{8}\right)}\sin\left(\frac{\beta\phi_0}{2}\right)\,. \end{aligned} \tag{11}$$

Zamolodchikov also gave the boundary energy as

$$\begin{aligned} E(\eta,\vartheta) = &- \frac{M}{2\cos\frac{\pi}{2\lambda}}\Big(\cos\left(\frac{\eta}{\lambda}\right)+\cosh\left(\frac{\vartheta}{\lambda}\right)-\frac{1}{2}\cos\left(\frac{\pi}{2\lambda}\right) \\ &+ \frac{1}{2}\sin\left(\frac{\pi}{2\lambda}\right)-\frac{1}{2}\Big)\,. \end{aligned} \tag{12}$$

The above formula for the boundary energy can be derived using the thermodynamic Bethe Ansatz for sinh-Gordon theory. The derivation, however, contains some nontrivial analytic continuation due to the fact that the sinh-Gordon TBA has no plateau solution in the ultraviolet limit [14]. For more details see [3], where we also gave a derivation of the UV-IR relation from the exact vacuum expectation values of boundary fields conjectured by Lukyanov, Zamolodchikov and Zamolodchikov [15].

We can perform a check on eqn. (12) which relates the boundary energy obtained from TBA to the bootstrap spectrum, thereby showing their consistency. It was noted in [9] (for Dirichlet boundary condition) and in [2] (for the general case) that continuing analytically

$$\eta \to \pi(\lambda+1)-\eta$$

the roles of the boundary ground state $|\rangle$ and of the boundary first excited state $|0\rangle$ are interchanged. Therefore we can calculate the energy difference between these two states from the formula for the boundary energy, eqn. (12). The result is

$$E(\pi(\lambda+1)-\eta,\vartheta)-E(\eta,\vartheta) = M\cos\left(\frac{\eta}{\lambda}-\frac{\pi}{2\lambda}\right)$$

which exactly equals the prediction of the bootstrap, i.e.

$$E_{|0\rangle}-E_{|\rangle} = M\cos\nu_0$$

that follows from eqn. (7). In our paper [3] we also checked that (12) was consistent with other information known from previous literature, like the boundary energy of the Lee-Yang model [16] and the boundary energy for sine-Gordon with Dirichlet boundary conditions [17].

4. TCSA verification

Truncated Conformal Space is a method to compute the spectrum of a perturbed CFT in finite volume. It works by computing the matrix elements of the Hamiltonian in the basis of the conformal states and then truncating the space of states to a finite dimensional subspace by imposing a cut-off in the conformal weight. It was introduced by Yurov and Zamolodchikov for the Lee-Yang model with periodic boundary conditions [18] and later extended to perturbations of $c = 1$ theories [19] and to theories with boundary [16]. As the boundary sine-Gordon model is a perturbed $c = 1$ CFT with boundary conditions, we could use these developments to compute the spectrum on a space-time strip, where space was an interval with (not necessarily identical) integrable boundary conditions on both ends. For a description of the technical details see [1].

On the other hand, one can make predictions for the finite size spectrum starting from the knowledge of the bootstrap spectrum and reflection factors and using a Bethe Ansatz technique.

The detailed results are described in [1] for the case of Neumann boundary condition ($M_0 = 0$ in eqn. (1)) and in [3] for Dirichlet ($M_0 \to \infty$) and general boundary conditions. Here we only describe the main conclusions.

1 The formulae (11,12) and the ground state reflection factors (3,5) are in very good agreement with the numerical data.

2 Boundary excited states can be obtained as analytic continuation of certain one-particle states in the framework of Bethe Ansatz [1]. The spectrum of the states that are accessible in TCSA matches precisely with the one conjectured from the bootstrap.

3 If a pole in the reflection factors was explained by some boundary Coleman-Thun diagram and has no corresponding bound state in the bootstrap, there is no such state in the TCSA spectrum either. This makes us confident that the rules used to draw these diagrams are indeed correct, which is very important as for the time being they have no field theoretical derivation.

5. Conclusions

We reported on work leading to a complete on-shell description of boundary sine-Gordon theory using bootstrap methods. We also derived Zamolodchikov's formulae for the boundary energy and the UV-IR relation and compared the results to numerical TCSA calculations. We found an excellent agreement and confirmed the general picture that was formed of boundary sine-Gordon theory in the previous literature.

The main open problems are the calculation of off-shell quantities (e.g. correlation functions) and exact finite size spectra. While correlation functions in general present a very hard problem even in theories without boundaries, in integrable theories significant progress was made using form factors (for one-point functions of bulk operators see e.g. [20]). In addition, the vacuum expectation values of boundary operators in sine-Gordon theory are also known exactly [15]. It would be interesting to make further progress in this direction.

Concerning finite size spectra, there is already a version of the so-called nonlinear integral equation for the vacuum (Casimir) energy with Dirichlet boundary conditions [17], but it is not yet clear how to extend it to describe excited states and more general boundary conditions as well, which also seems to be a fascinating problem.

It would also be interesting to work out a formalism (an analogue of the Cutkosky rules of quantum field theory in the bulk) in which the rules for the boundary Coleman-Thun diagrams can be justified.

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Notes

1. A similar relation was derived by Corrigan and Taormina [12] for sinh-Gordon theory, however, their normalization of the coupling constants is different from the one natural in the perturbed CFT framework.

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