Finite Size Effects in sine-Gordon model

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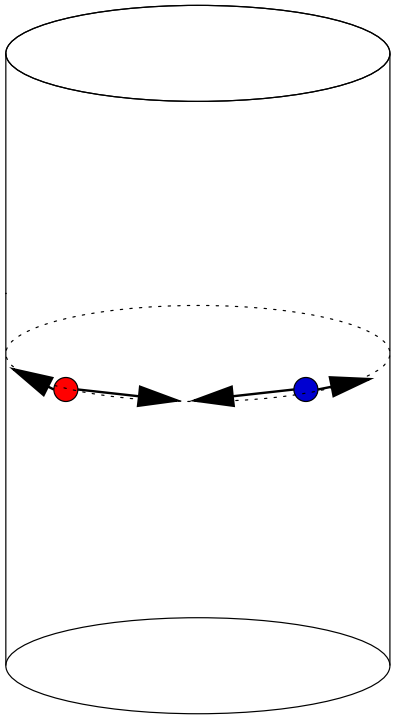
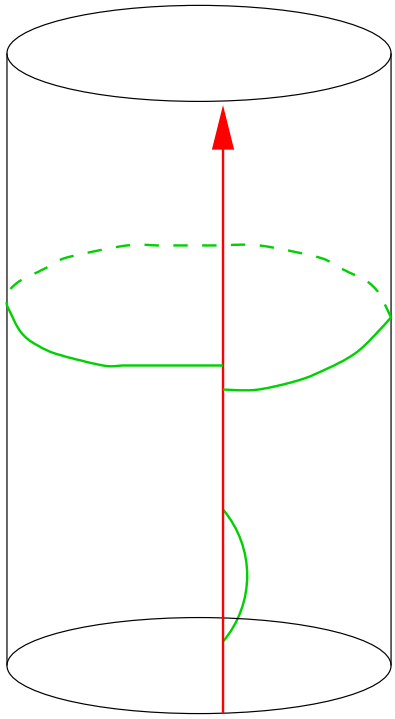
Finite Size Effects

Casimir Effect: Particles feel direct interaction but also other side of the cylinder one. In QFT new radiative corrections may appear on closed geometry.





QFT: Lüscher has shown how to get S-matrix and other info from finite size effects

In 2D QFT we are interested in the spectrum evolution under RG flows $\Longrightarrow$ **Scaling Functions**

Scaling Functions

Casimir Energy of the i-th excited state on a cylinder of circumference L

$$E_i = -\frac{\pi\, c_i(l)}{6L}$$

where $l = ML$ is dimensionless and M is a mass scale or mass gap.
$c_i(l)$ are the **scaling functions.**

- $l \to 0$ gives UV limit $\Longrightarrow$ CFT data

$$c_i(0) = c - 12(\Delta_i + \bar{\Delta}_i)$$

- $l \to \infty$ gives IR limit $\Longrightarrow$ S-matrix data & mass gaps

Computation of Scaling Functions

- **CPT** = Conformal Perturbation Theory (A.Zam. 85 - Cardy Ludwig 86)

$$c_i(l) = c_i(0) + Bl^2\,(+\tilde{B}l^2 \log l) + \sum_{k=0}^{\infty} C_i^{(k)}\, l^{\,k(2-2\Delta_{pert})+1}$$

does not catch non-perturbative effects and works only very deep in UV.

- **TCSA** = Truncated Conformal Space Approach (Yurov Al.Zam. 90): diagonalize the truncated ($E < E_{cut}$) Hamiltonian $H_{CFT} + V_{pert}$ numerically.
  Non-perturbative, but affected by **truncation errors** and we have **no analytic control (UV divergences!).**

- **TBA** = Thermodynamic Bethe Ansatz (Al. Zam. 90): thermodynamics of a gas of particles governed by the given **S-matrix**.
  Free Energy $\Longrightarrow$ Finite Size vacuum Energy
  Results in a set of coupled **NLIEs** for Free Energy.
  Only **vacua** (with periodic or twisted b.c.) are accessible.

- **DdV** = Destri De Vega NLIE from light cone Bethe ansatz (Destri DeVega 92-95, Klümper Batchelor Pearce 91-93-95) .
  Only **vacua** (at the beginning) and only Sine-Gordon.
  It must be **proven** that the lattice construction really gives sG/mTh.

The main criticism is: only few states (vacua) accessed by the exact methods, while all excited states accessed by TCSA, but with truncation errors.

However....

The "Exciting" Summer 1996

- **Bazhanov Lukyanov A.Zam.** obtain a **conjecture** for the 1st excited state TBA eq. in SLYM from their QISM formulation of CFT.

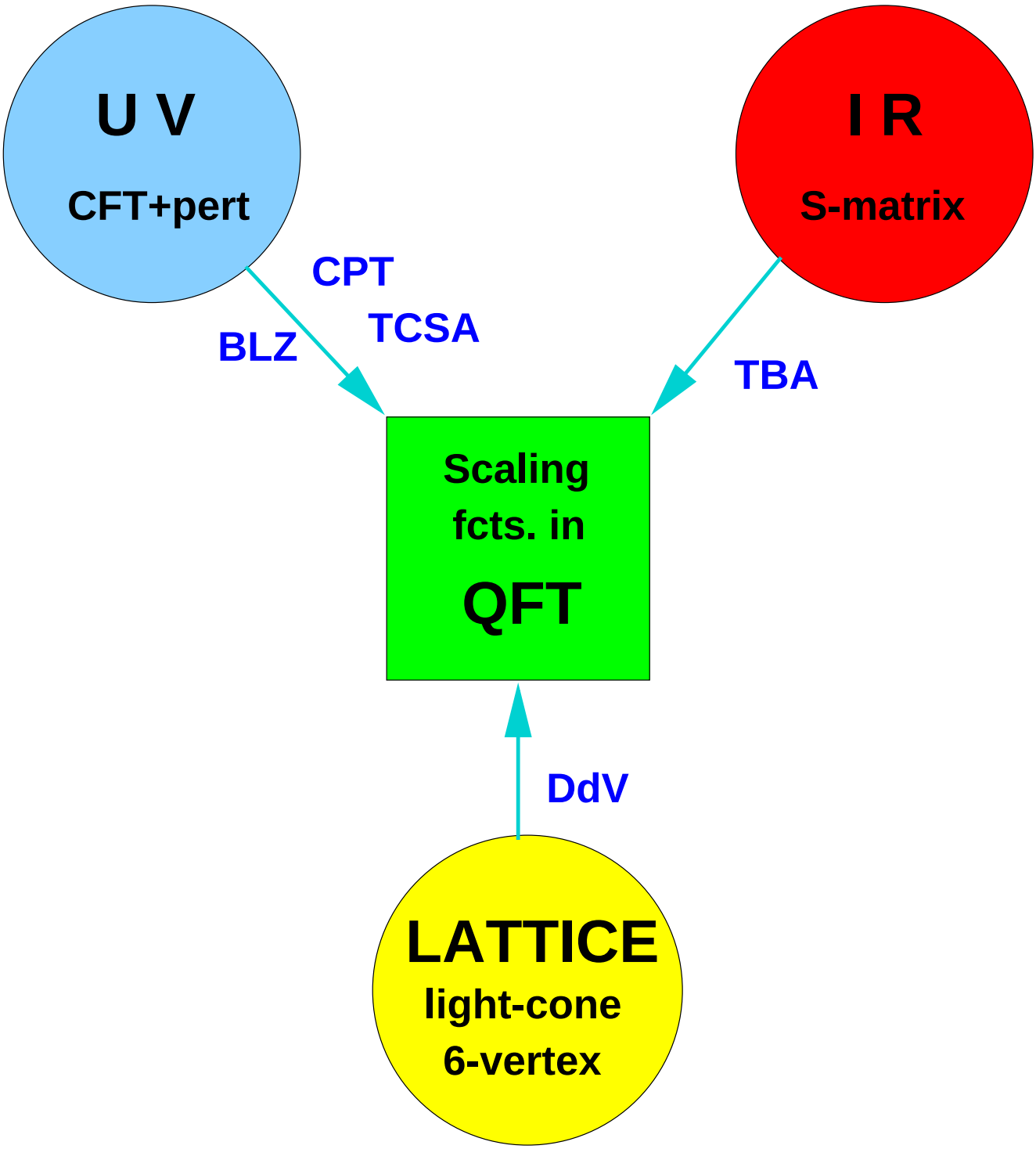
- **Dorey Tateo** continue analytically in l the TBA vacuum scaling function of SLYM, turn around a branch singularity and come back to real l in a new Riemann determination $\Longrightarrow$ **TBA for excited states.**

- **Fioravanti Mariottini Quattrini Ravanini** create excited configurations of Bethe roots and get the **excited state DdV NLIE.**

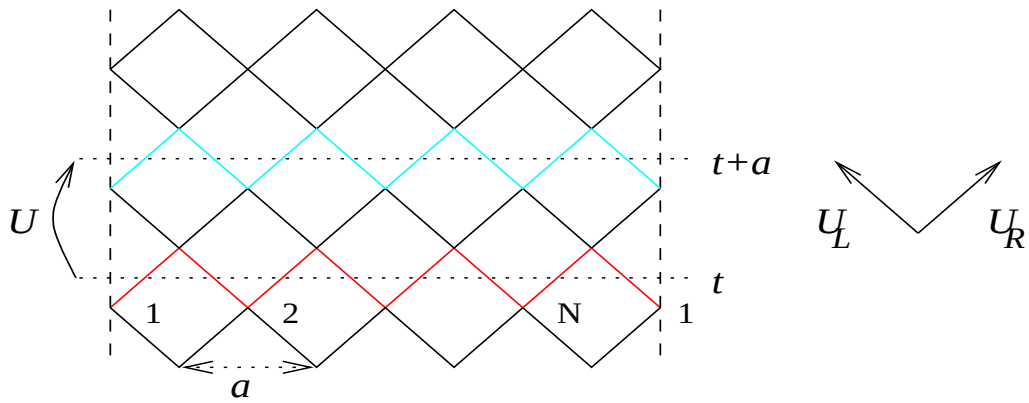
- Feb 1997 **Destri De Vega** generalize the Bologna group result to the most **general even charge** excited state.

- Mar-May 1998 **Feverati Ravanini Takács** develop TCSA for c=1 CFT and thus establish the correct **quantization rule** for general even charge sG states in DdV.

- Jul 1998 **Feverati Ravanini Takács** conjecture and check against TCSA a generalization of DdV to the odd charge sector $\Longrightarrow$ **control on the full sG Hilbert space.**


U V
CFT+pert
I R
S-matrix
CPT
TCSA
BLZ
TBA
Scaling
fcts. in
QFT
DdV
LATTICE
light-cone
6-vertex

Light-cone Lattice Bethe Ansatz

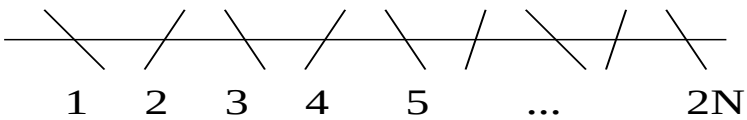


2-dim. space-time discretised along the **light-cone** with **periodic b.c.** in the spatial direction. U_L and U_R are the light-cone evolution operators, a is the lattice spacing, and the spatial volume is given by $L = Na$.

Unitary time evolution operator

$$\begin{aligned} U &= U_L U_R = e^{-iaH} \\ &= \tau^{(2N)}(\vartheta/2|\{\Theta_n\})\tau^{(2N)}(-\vartheta/2|\{\Theta_n\})^{-1} \end{aligned}$$

where $\tau^{(2N)}(\vartheta|\{\Theta_n\})$ is the 6-vertex transfer matrix with anisotropy γ on a square lattice of $2N$ sites with inhomogeneities



which we choose to be $\{\Theta_n = (-1)^n\Theta\}$.

Diagonalization of transfer matrix: Bethe Ansatz Equations (BAE)

$$\left(\frac{\sinh\frac{\gamma}{\pi}\left[\vartheta_j+\Theta+\frac{i\pi}{2}\right]\sinh\frac{\gamma}{\pi}\left[\vartheta_j-\Theta+\frac{i\pi}{2}\right]}{\sinh\frac{\gamma}{\pi}\left[\vartheta_j+\Theta-\frac{i\pi}{2}\right]\sinh\frac{\gamma}{\pi}\left[\vartheta_j-\Theta-\frac{i\pi}{2}\right]}\right)^N = -\prod_{k=1}^{M}\frac{\sinh\frac{\gamma}{\pi}\left[\vartheta_j-\vartheta_k+i\pi\right]}{\sinh\frac{\gamma}{\pi}\left[\vartheta_j-\vartheta_k-i\pi\right]}$$

ϑ_j: roots, their number is $M \leq N$.

Energy and momentum

$$e^{ia(E\pm P)/2} = (-1)^M \prod_{j=1}^{M}\frac{\sinh\frac{\gamma}{\pi}\left[\Theta\pm\vartheta_j+\frac{i\pi}{2}\right]}{\sinh\frac{\gamma}{\pi}\left[\Theta\pm\vartheta_j-\frac{i\pi}{2}\right]} + O(1)$$

Counting function

$$Z_N(\vartheta) = N[\phi_{1/2}(\vartheta+\Theta)+\phi_{1/2}(\vartheta-\Theta)] - \sum_{j=1}^{M}\phi_1(\vartheta-\vartheta_k)$$

where $\phi_\nu(\vartheta) = i\log\frac{\sinh\frac{\gamma}{\pi}(i\pi\nu+\vartheta)}{\sinh\frac{\gamma}{\pi}(i\pi\nu-\vartheta)}$.

Bethe Ansatz Equations with counting function

$$Z_N(\vartheta_j) = 2\pi I_j\,, \quad I_j \in \mathbb{Z}+\frac{1+\delta}{2}\,, \quad \delta = M \bmod 2\,.$$

Root classification

Real roots: $\vartheta_j \in \mathbb{R}$.

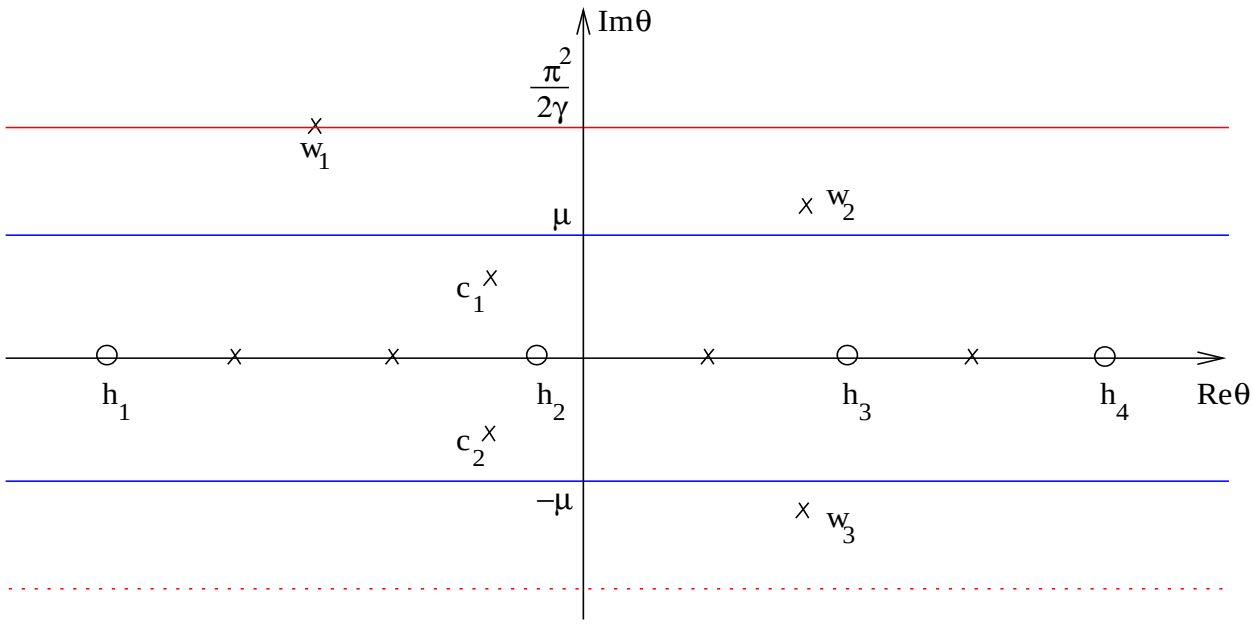
Holes: locations $h_k \in \mathbb{R}$ where $Z_N(h_k) = 2\pi I_k$, but are not in the set of real roots.

Close complex roots: $0 < |\Im m\, c_j| < \min\left\{\pi,\ \pi\left(\frac{\pi}{\gamma} - 1\right)\right\}$, in complex conjugate pairs.

Nonselfconjugate wide roots: $\mu = \min\left\{\pi,\ \pi\left(\frac{\pi}{\gamma} - 1\right)\right\} < |\Im m\, w_j| < \frac{\pi^2}{2\gamma}$, in complex conjugate pairs.

Selfconjugate wide roots: $|\Im m\, w_j| = \frac{\pi^2}{2\gamma}$.

Special roots/holes: real roots/holes y_k where $Z'_N(y_k) < 0$.



The Destri-de Vega Equation

In the continuum limit $N \to \infty$, $a \to 0$, L fixed and $\Theta \sim \log \frac{4N}{\mathcal{M}L}$, using a residue trick, one can write a single **nonlinear integral equation** for the counting function:

$$\begin{aligned} Z(\vartheta) = & \quad l \sinh \vartheta + \alpha + g(\vartheta|\vartheta_j) + \\ + & \quad 2\Im m \int_{-\infty}^{\infty} dx G(\vartheta - x - i\eta) \log \left(1 + (-1)^{\delta} e^{iZ(x+i\eta)}\right) \end{aligned}$$

Bethe quantisation: $Z(\vartheta_j) = 2\pi I_j$, with source term:

$$\begin{aligned} g(\vartheta|\vartheta_j) = & \quad \sum_{k=1}^{N_H} \chi(\vartheta - h_k) - 2\sum_{k=1}^{N_S} \chi(\vartheta - y_k) - \\ - & \quad \sum_{k=1}^{M_C} \chi(\vartheta - c_k) - \sum_{k=1}^{M_W} \chi(\vartheta - w_k)_{II} \end{aligned}$$

where

$$G(\vartheta) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} dk\, e^{ik\vartheta} \frac{\sinh \frac{\pi(p-1)k}{2}}{2 \sinh \frac{\pi p k}{2} \cosh \frac{\pi k}{2}}, \; p = \frac{\pi}{\gamma} - 1$$

$$\chi(\vartheta) = 2\pi \int_0^{\vartheta} dx G(x)$$

$$\chi(\vartheta)_{II} = \begin{cases} \chi(\vartheta) + \chi\left(\vartheta - i\pi \mathrm{sign}\left(\Im m \vartheta\right)\right), & p > 1, \\ \chi(\vartheta) - \chi\left(\vartheta - ip\pi \mathrm{sign}\left(\Im m \vartheta\right)\right), & p < 1. \end{cases}$$

Energy

$$\begin{aligned} E = & \quad E_{bulk} + \mathcal{M} \sum_{j=1}^{N_H} \cosh h_j - 2\mathcal{M} \sum_{j=1}^{N_S} \cosh y_j - \\ - & \quad \mathcal{M} \sum_{j=1}^{M_C} \cosh c_j + \mathcal{M} \sum_{j=1}^{M_W} (\cosh w_j)_{II} \\ - & \quad \mathcal{M} \int_{-\infty}^{\infty} \frac{dx}{2\pi} 2 \Im m \left[\sinh(x+i\eta) \log(1+(-1)^{\delta} e^{iZ(x+i\eta)}) \right] \end{aligned}$$

Infrared limit: $l \to \infty$, the convolution term vanishes exponentially fast $\Longrightarrow$

Dressed asymptotic Bethe ansatz

$$Z(\vartheta_i) = l \sinh(\vartheta_i) - i \log S(\vartheta_i - \vartheta_j) = 2\pi I_i$$

One reconstructs the following **sG S-matrix** eigenvalues:

Two holes $\Longrightarrow$ two solitons of the same charge

Two holes + complex pair $\Longrightarrow$ soliton-antisoliton, antisymmetric

Two holes + selfconjugate $\Longrightarrow$ soliton-antisoliton, symmetric

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Conjecture: the lattice model gives an integrable regularisation of sine-Gordon theory.

Analysis of the UV spectrum

UV limit of sG theory: **c=1 free boson CFT**: operator algebra generated by vertex operators $V_{(n,m)}(x,t)$ with conformal weights

$$\Delta^{\pm} = \frac{1}{2}\left(\frac{n}{R} \pm \frac{1}{2}mR\right)^2 \quad , \quad R = \frac{\sqrt{4\pi}}{\beta} = \sqrt{\frac{1}{2}\left(1+\frac{1}{p}\right)}$$

for sG $n, m \in \mathbb{Z}$.

sG is obtained by perturbing the UV fixed point with the operator

$$V = \lambda \int \left(V_{(1,0)}(x,\ t) + V_{(-1,0)}(x,\ t)\right) dx \qquad \lambda = \frac{m_0^2}{\beta^2}\,.$$

UV dimensions predicted by DdV equation:

1. The **vacuum** state: on the lattice $M = N$ and all Bethe roots are real. There are two possibilities: $\delta = 0$ or 1.

$$\begin{array}{lll} \delta = 0: & \Delta^{\pm} = 0 & \text{i.e. } 1 \\ \delta = 1: & \Delta^{\pm} = \frac{1}{8R^2} & \text{i.e. } V_{(\pm 1/2,0)} \end{array}$$

   i.e. the physical vacuum is the one with $\delta = 0$.

2. The **two-soliton** (two-hole) state

   (a) $\delta = 0$: $I_1 = -I_2 = \frac{1}{2} \Longrightarrow \Delta^{\pm} = \frac{R^2}{2}$ i.e. $V_{(0,2)}$.
   (b) $\delta = 1 \Longrightarrow$ a $V_{(\pm 1/2,2)}$ descendent, not in UV sG spectrum.

3. The **symmetric soliton-antisoliton** state (two holes + a self-conjugate root)
   $\delta = 1$, $I_1 = -I_2 = 1$ and $I_c^{\pm} = 0 \Longrightarrow \Delta^{\pm} = \frac{1}{2R^2}$ i.e. $V_{(\pm 1,0)}$

4. The **antisymmetric soliton-antisoliton** state (two holes + a complex pair)
   $\delta = 0$, $I_1 = I_c^{-} = -I_2 = -I_c^{+} = \frac{1}{2} \Longrightarrow \Delta^{\pm} = \frac{1}{2R^2}$ i.e. $V_{(\pm 1,0)}$

To obtain states which are present in the UV spectrum of sine-Gordon theory, the quantisation rule should be chosen according to

$$\delta = M_{sc} \bmod 2$$

where M_{sc} is the number of selfconjugate roots.

The “odd” conjecture

S =spin of the lattice XXZ chain. Light-cone lattice gives:

$$m = 2S$$
$$2n = (\delta + 2S + M_{sc}) \mod 2$$

$S \in \mathbb{Z} \Rightarrow$ access only to even topological charge sectors $Q = m$

Conjecture: S half-integer gives the odd charge sectors.

sG with periodic b.c. needs $n \in \mathbb{Z} \Rightarrow$ Quantization rule

$$(\delta + M_{sc}) \mod 2 = 1.$$

Counting equation

$$N_H - 2N_S = 2S + M_C + 2\theta(\pi - 2\gamma)M_W \ ,$$

implies that if $S \in \mathbb{Z} + \frac{1}{2}$ then N_H must be odd $\Rightarrow$ odd number of hole sources

Attention: Not deducible from light-cone lattice: **conjecture** (new fermionic light-cone lattice formulation) (Destri-Segalini 95)

Support 1: Numeric comparison with TCSA (see below)

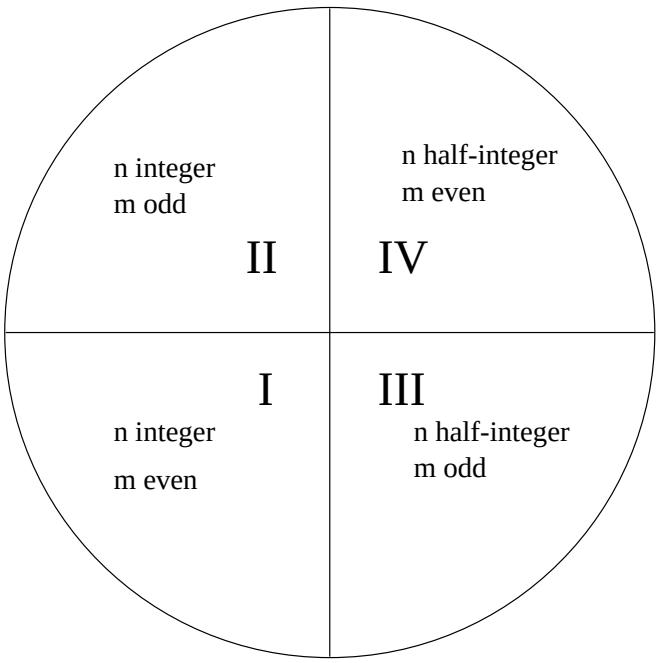
Support 2: Examples of UV dimensions computed from DdV.

One hole, $I = 0,\ \delta = 1 \Longrightarrow \Delta^{\pm} = \frac{1}{8R^2}$ i.e. vertex op. $V_{(0,1)}$.

Three holes, $I_1 = 1,\ I_2 = 0,\ I_3 = -1 \Longrightarrow$ vertex op. $V_{(0,3)}$.

Higher values of quantum numbers $\Longrightarrow V_{(n,3)},\ n \in \mathbb{Z}$
Three holes + close pair $\Longrightarrow$ 1st nonchiral desc. of $V_{(0,1)}$.

"Odd" conjecture: sG states with odd topological charge correspond to taking an odd number of holes and choosing the quantisation rule $\delta + M_{sc} = 1 \bmod 2$.



Sine-Gordon = **I + II** Thirring = **I + III** **IV** = non-local
"even" DdV $\Rightarrow$ states in **I + IV**, extended DdV $\Rightarrow$ all sectors
Quantization rule $\Leftrightarrow$ choice of local projection

α-sectors

Klassen-Melzer 92: for generic R only two local projections in UV:

$$\mathcal{A}_b = \bigoplus_{n,m\in\mathbb{Z}} [V_{n,m}]$$
$$\mathcal{A}_f = \bigoplus_{\substack{m\in\mathbb{Z} \\ n\in\mathbb{Z}+\frac{m}{2}}} [V_{n,m}]$$

Choosing $V_{1,0} + V_{-1,0}$ as perturbation $\Rightarrow$ sG/mTh.

Choosing $V_{k,0} + V_{-k,0}$: R can be redefined to get perturbing operator $V_{1,0} + V_{-1,0}$ and the algebras

$$\mathcal{A}_b^{(k)} = \bigoplus_{n,m\in\mathbb{Z}} \left[V_{\frac{n}{k},km}\right]$$
$$\mathcal{A}_f^{(k)} = \bigoplus_{\substack{m\in\mathbb{Z} \\ n\in\mathbb{Z}+\frac{m}{2}}} \left[V_{\frac{n}{k},km}\right]$$

States in these sectors can be realised by α-twists $\frac{2\pi r}{k}$.

Rational compactifications allow for other local projections:

$$R = \sqrt{\frac{1}{2}\left(1 + \frac{1}{p}\right)}, \quad p = 3, 4, \ldots$$

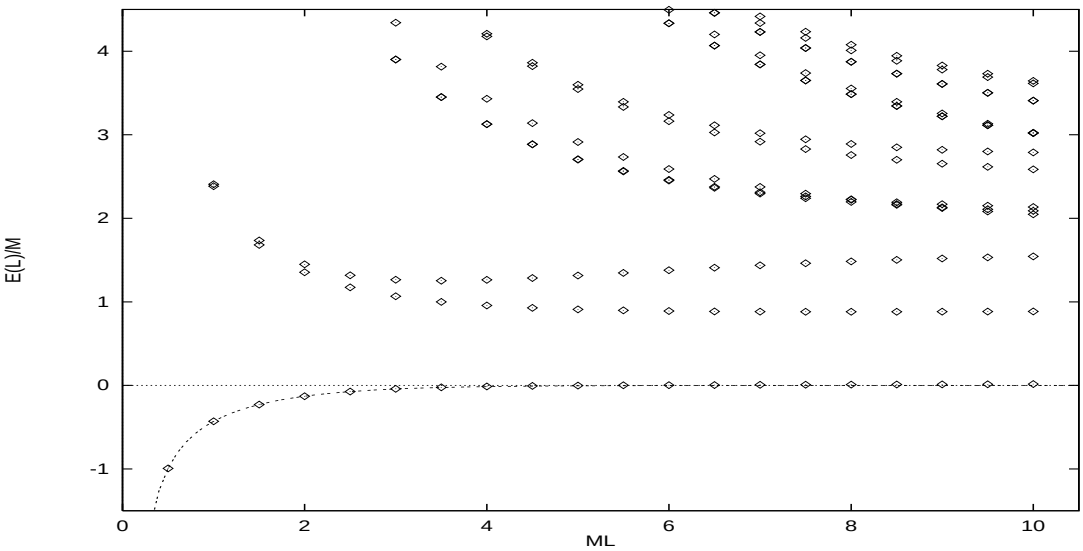
and $\alpha = \frac{k\pi}{p}, \quad k = 1, \ldots, p-1 \Rightarrow$ Hilbert space $\mathcal{H}_p$ has a local projection onto $\mathcal{M}_{p,p+1} + \Phi_{1,3}$.

In this theory, **kinks** are created in UV limit by $\phi_{2,1}$.

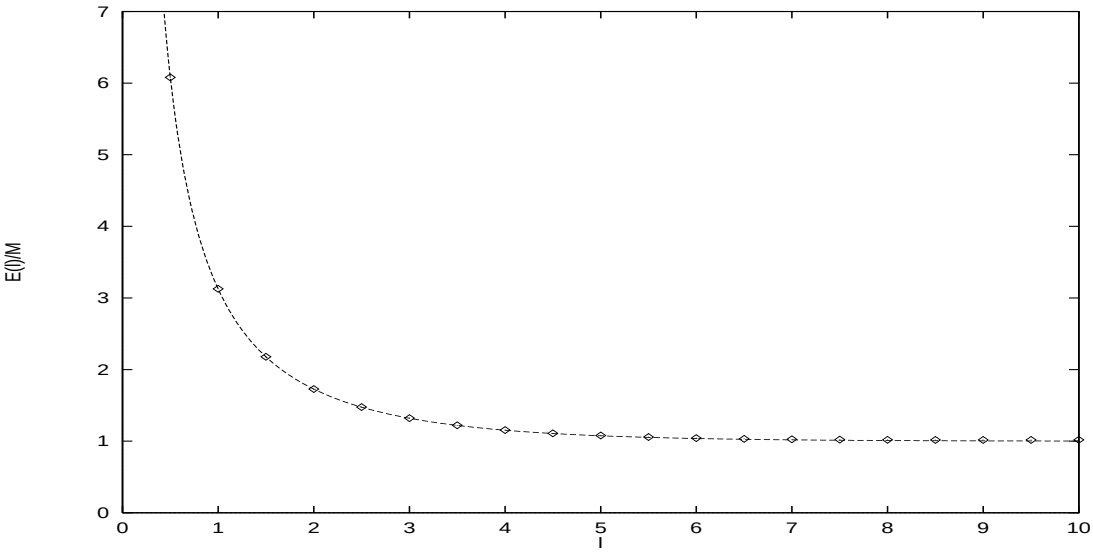
$p \to \infty$ (marginal point $\beta^2 = 8\pi$): **SU(2) Gross-Neveu**

- identical to local projection $\mathcal{A}_b^{(2)}$
- $\phi_{2,1}$ tends to an operator with UV dim. 1/4 (consistent with Gross-Neveu kinks)

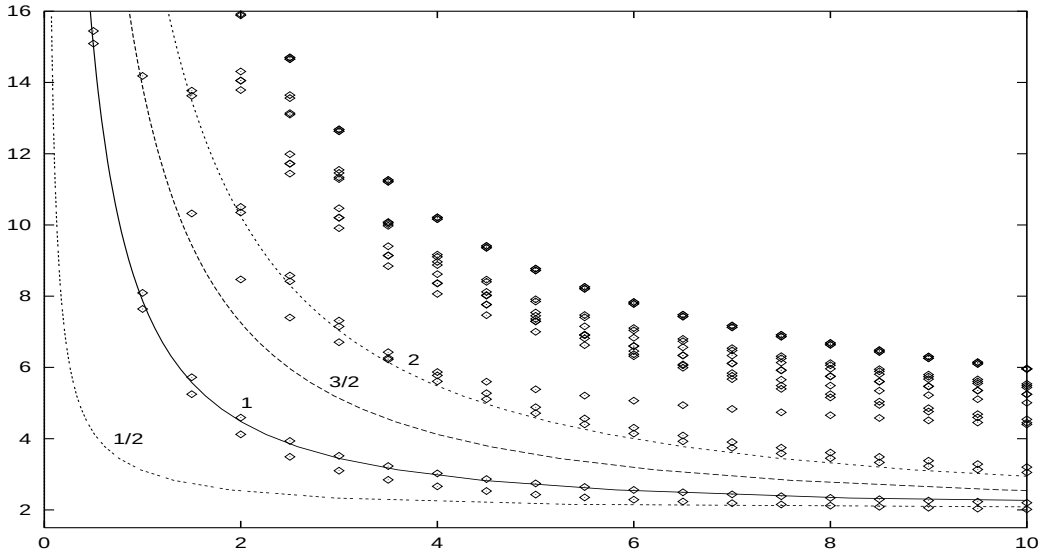
Numerical comparison between TCS and DdV



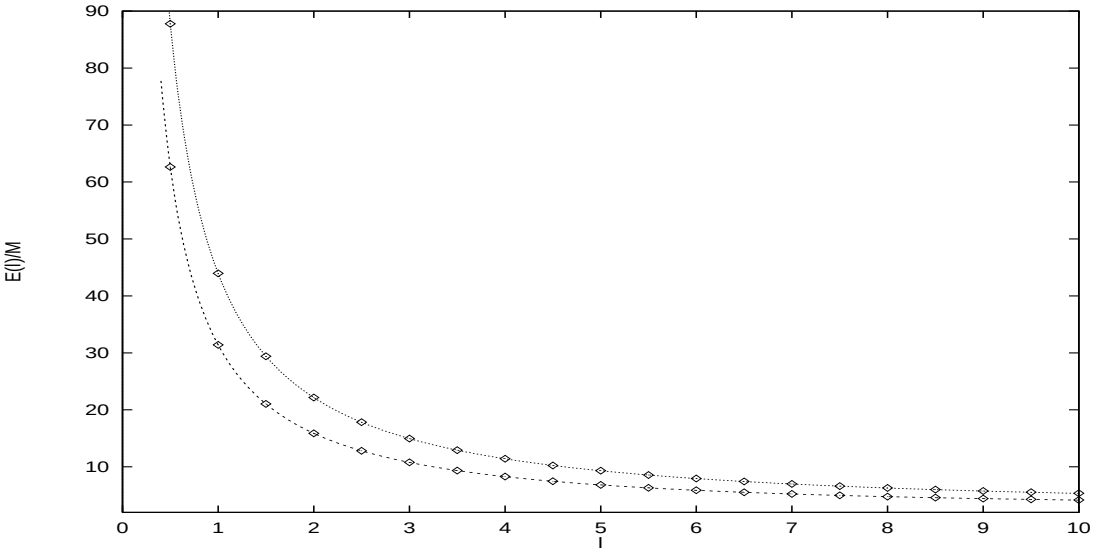
Vacuum energy at $p = 2/7$



One soliton energy at $p = 2/7$



Soliton-antisoliton states in the parity-even channel at $p = 1.5$



The two lowest lying states with 3 solitons of the same charge, at $p = 2/7$

Conclusions

1. The Destri-de Vega equation does describe the excited state scaling functions of the sine-Gordon / massive Thirring theory.

2. Open questions:

   (a) Do the scaling functions obtained from the DdV equation form a complete set?
   (b) Relation to the description of excited states in perturbed rational conformal field theories by analytic continuation of Thermodynamical Bethe Ansatz.
   (c) To perform the local projection on $\mathcal{M}_{p,p+1} + \Phi_{13}$ and obtain the scaling functions.

3. There exists an A, D, E generalization. How to extend it to nonsimply-laced algebras (esp. ZMS model and Φ_{12}, Φ_{21}, Φ_{15} perturbations)?

This presentation is based on the results obtained in the papers:
hep-th/9608091, hep-th/9803104,
hep-th/9805117, hep-th/9807160.