Current noise in mesoscopic conducting systems

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Mean square fluctuations

- The noise is characterized by the mean square fluctuations:

$$\langle \delta I^2 \rangle = \left\langle \left(I - \langle I \rangle \right)^2 \right\rangle$$

Or in an alternate form:

$$\langle \delta I^2 \rangle = \langle I^2 \rangle - \langle I \rangle^2$$

Where „I" stands for the instant value of the current, „<I>" is the average value and „$\langle I^2 \rangle$" for the current square average value.

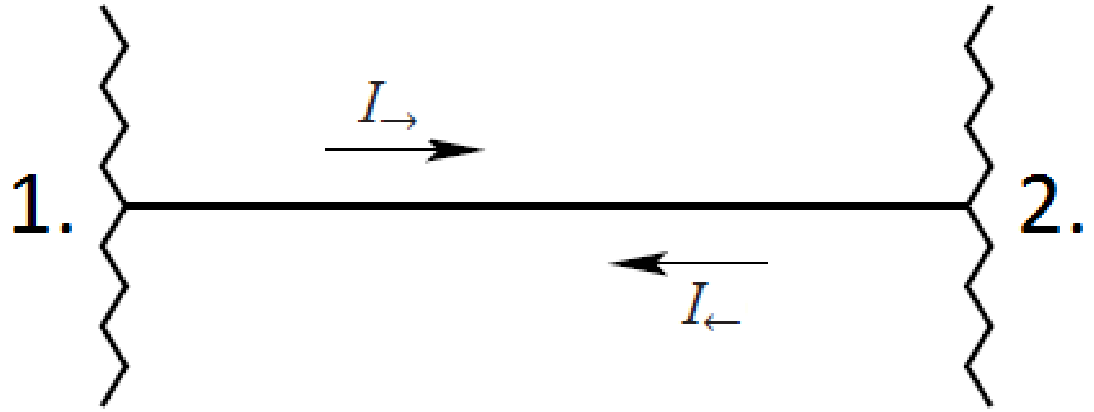
Different types of noise

- Thermal noise
  - - Exists even in equilibrium
  - - Caused by the fluctuations in the occupation number
  - - This kind of noise is absent at zero temperature

- Shot noise
  - - Arises only in non-equilibrium
  - - Caused by the probabilistic propagation of electrons
  - - Exists even at zero temprature
  - - If the scattering is deterministic (T=1 or R=1), this kind of noise is absent

Detailed discussion: Thermal noise

- Simplified model:
  - - The sample connects two reservoirs
  - - Electron with E energy propagate ballistically ($T_{12}(E) = T_{21}(E) = 1$)
  - - Electrons with $E' \neq E$ do not propagate ($T_{12}(E') = T_{21}(E') = 0$)
  - - Directions:
    - - Right: from 1. reservoir to 2. reservoir
    - - Left: from 2. reservoir to 1. reservoir



Calculating the thermal noise

- Current:
  - - We can calculate the current: $\langle I_{\rightarrow} \rangle = I_0 \, P_{\rightarrow}$
    - - $I_0 = ev/\mathfrak{L}$
      - - „e” - electron charge
      - - „v” - electron velocity
      - - „$\mathfrak{L}^{-1}$” - electron density
      - - „$P_{\rightarrow}$” – probability of the occupation of the corresponding state
  - The probability equals to the Fermi-function

$$P_{\rightarrow} = f_1(E)$$

Calculating the thermal noise

The probability can be defined as the ratio of the time when the state is occupied and the total (observation) time.

$$P_{\rightarrow} = \lim_{\mathcal{T}\to\infty} \frac{\Delta t_{\rightarrow}}{\mathcal{T}}$$

The mean current can be calculated:

$$\langle I_{\rightarrow} \rangle = \lim_{\mathcal{T}\to\infty} \frac{1}{\mathcal{T}} \int_0^{\mathcal{T}} dt\, I_{\rightarrow}(t) = \lim_{\mathcal{T}\to\infty} \frac{I_0 \Delta t_{\rightarrow}}{\mathcal{T}} = I_0 P_{\rightarrow}$$

$$\downarrow$$

$$\langle I_{\rightarrow}^2 \rangle = \lim_{\mathcal{T}\to\infty} \frac{1}{\mathcal{T}} \int_0^{\mathcal{T}} dt\, I_{\rightarrow}^2(t) = \lim_{\mathcal{T}\to\infty} \frac{I_0^2 \Delta t_{\rightarrow}}{\mathcal{T}} = I_0^2 P_{\rightarrow}$$

Calculating the thermal noise

From these two equations we get: $\langle \delta I_{\rightarrow}^2 \rangle = I_0^2 P_{\rightarrow}\left(1 - P_{\rightarrow}\right)$

If $P_{\rightarrow}$= 0 or $P_{\rightarrow}$=1, then there is no thermal noise.

If 0 <$P_{\rightarrow}$< 1, then the thermal noise exists with maximum at $P_{\rightarrow}$ = 0.5

Using the Fermi-function's property: $f_1(E)\left(1 - f_1(E)\right) = \left(-\frac{\partial f_1(E)}{\partial E}\right) k_B T_1$

$$\downarrow$$

$$\langle \delta I_{\rightarrow}^2 \rangle = I_0^2 \left(-\frac{\partial f_1(E)}{\partial E}\right) k_B T_1$$

Calculating the thermal noise

Now calculating with both direction:

$$\langle I \rangle = \langle I_{\rightarrow} \rangle - \langle I_{\leftarrow} \rangle = I_0 \left\{ f_1(E) - f_2(E) \right\}$$

Assuming, that the electrons from the two reservoirs are uncorrelated:

$$\langle \delta I^2 \rangle = \langle \delta I^2_{\rightarrow} \rangle + \langle \delta I^2_{\leftarrow} \rangle = I_0^2 \left[f_1(E) \left\{ 1 - f_1(E) \right\} + f_2(E) \left\{ 1 - f_2(E) \right\} \right]$$

If the two reservoirs have the same Fermi-function and temperature, we get:

$$\langle I \rangle = 0$$

$$\langle \delta I^2_{\rightarrow} \rangle = 2 I_0^2 \left(-\frac{\partial f_0(E)}{\partial E} \right) k_B T_0$$

Detailed discussion: Shot noise

-Simplified model:

- - The sample connects two reservoirs
- - The transmission factor in both directions are equal $T_{12}(E) = T_{21}(E)$
- - A bias is applied to the system, so the electrons can come only from the first reservoir

$$\mu_2 + eV_2 < E < \mu_1 + eV_1 \Rightarrow f_1(E) = 1,\, f_2(E) = 0$$

- - We use the ballistic model

$$I_0 = ev/\mathfrak{L}$$

$$\Big\downarrow \quad T_{21} = \lim_{\mathfrak{T}\to\infty} \frac{\Delta t_{\to}}{\mathfrak{T}}$$

$$\langle I \rangle = I_0\, T_{21}(E)$$

$$\langle \delta I^2 \rangle = I_0 \langle I \rangle \left\{ 1 - T_{21}(E) \right\}$$

Detailed discussion: Mixed noise

Now we consider a case when both noises are present.

- Our model:
  - - The scatterer is connected to two reservoirs
  - - The reservoirs have different temperature
  - - The reservoirs have different potentials

- The current:
  - - In this case the current depends on 2 things:
    - - The occupation probability of the corresponding state
    - - The transmission probability of the sample

$$P_{\leftarrow} = T_{12}(E)\, f_2(E) \qquad\qquad P_{\rightarrow} = T_{21}(E)\, f_1(E)$$

Calculating the mixed noise

The current flowing through the system:

$$\langle I \rangle = I_0 \, T_{12}(E) \left\{ f_1(E) - f_2(E) \right\}$$

The noise is not easy to calculate, because the currents in different directions are correlated. Two electrons can't occupy the same state, because of the Pauli exclusion principle.

$\downarrow$

If one state is occupied, the electron coming from a different reservoir can not occupy the same state.

$\downarrow$

The fluctuations in the current become correlated. $\longrightarrow$ $\langle \delta I^2 \rangle \neq \langle \delta I_{\rightarrow}^2 \rangle + \langle \delta I_{\leftarrow}^2 \rangle$

Calculating the mixed noise

-We have to describe the electrons quantum-mechanically.

- - We use second-quantization formalis
- - Creation /annihilation operators:
  - - Incident from 1,2 reservoir: $\hat{a}_\alpha^\dagger / \hat{a}_\alpha$
  - - Scattered into 1,2 reservoirs: $\hat{b}_\alpha^\dagger / \hat{b}_\alpha$

- We use the already known scattering matrix and the operator’s other form:

$$\hat{b}_\alpha = \sum_{\beta=1}^{2} S_{\alpha\beta} \hat{a}_\beta , \quad \hat{b}_\alpha^\dagger = \sum_{\beta=1}^{2} S_{\alpha\beta}^* \hat{a}_\beta^\dagger$$

Calculating the mixed noise

We calculate the current on the left side of the scatterer, and choose the positive direction from the scatterer to the reservoir. In this case the current operator:

$$\hat{I}_1 = I_0(\hat{b}_1^\dagger \hat{b}_1 - \hat{a}_1^\dagger \hat{a}_1)$$

We also use the particle number density operator:

$$\hat{n}_\alpha = \hat{a}_\alpha^\dagger \hat{a}_\alpha$$

After statistical averaging of the density operatoroperator, we get the Fermi function, and considering, that the electrons in different reservoirs are uncorrelated, we get:

$$\langle \hat{a}_\alpha^\dagger \hat{a}_\beta \rangle = \delta_{\alpha\beta} f_\alpha \,, \quad f_\alpha = \frac{1}{1 + e^{\frac{E-\mu_\alpha}{k_B T_\alpha}}} \,, \quad \alpha = 1, 2$$

Calculating the mixed noise

First we calculate the current:

$$\langle \hat{I}_1 \rangle = I_0 \langle \hat{b}_1^\dagger \hat{b}_1 - \hat{a}_1^\dagger \hat{a}_1 \rangle = I_0 \left\langle \sum_{\beta=1}^{2} S_{1\beta}^* \hat{a}_\beta^\dagger \sum_{\gamma=1}^{2} S_{1\gamma} \hat{a}_\gamma - \hat{a}_1^\dagger \hat{a}_1 \right\rangle$$

$$= I_0 \left\{ \sum_{\beta=1}^{2} \sum_{\gamma=1}^{2} S_{1\beta}^* S_{1\gamma} \langle \hat{a}_\beta^\dagger \hat{a}_\gamma \rangle - \langle \hat{a}_1^\dagger \hat{a}_1 \rangle \right\} = I_0 \left\{ \sum_{\beta=1}^{2} |S_{1\beta}|^2 f_\beta - f_1 \right\}$$

And the anticommutation relations: $\hat{a}_\alpha^\dagger \hat{a}_\beta + \hat{a}_\beta \hat{a}_\alpha^\dagger = \delta_{\alpha\beta}$

Calculating the mixed noise

Using the unitarity of the scattering matrix, and using the transmission probabilities:

$$\langle \hat{I}_1 \rangle = I_0 T_{12} (f_2 - f_1)$$

Now we calculate the mean square current fluctuations:

- - First we write the current operator in terms of incident electrons:

$$\hat{b}_1 = S_{11}\hat{a}_1 + S_{12}\hat{a}_2 \,, \qquad \hat{b}_1^\dagger = S_{11}^*\hat{a}_1^\dagger + S_{12}^*\hat{a}_2^\dagger$$

$$\hat{I}_1/I_0 = \hat{b}_1^\dagger \hat{b}_1 - \hat{a}_1^\dagger \hat{a}_1 = \left(S_{11}^*\hat{a}_1^\dagger + S_{12}^*\hat{a}_2^\dagger \right) (S_{11}\hat{a}_1 + S_{12}\hat{a}_2) - \hat{a}_1^\dagger \hat{a}_1 =$$

$$= T_{12}(\hat{a}_2^\dagger \hat{a}_2 - \hat{a}_1^\dagger \hat{a}_1) + S_{11}^* S_{12} \hat{a}_1^\dagger \hat{a}_2 + S_{12}^* S_{11} \hat{a}_2^\dagger \hat{a}_1 \,.$$

Calculating the mixed noise

We expand the square of current operator:

$$\hat{I}_1^2/I_0^2 = \left(T_{12}(\hat{a}_2^\dagger\hat{a}_2 - \hat{a}_1^\dagger\hat{a}_1) + S_{11}^*S_{12}\hat{a}_1^\dagger\hat{a}_2 + S_{12}^*S_{11}\hat{a}_2^\dagger\hat{a}_1\right)^2 =$$

$$= T_{12}^2\left(\hat{a}_2^\dagger\hat{a}_2\hat{a}_2^\dagger\hat{a}_2 + \hat{a}_1^\dagger\hat{a}_1\hat{a}_1^\dagger\hat{a}_1 - \hat{a}_2^\dagger\hat{a}_2\hat{a}_1^\dagger\hat{a}_1 - \hat{a}_1^\dagger\hat{a}_1\hat{a}_2^\dagger\hat{a}_2\right)$$

$$+R_{11}T_{12}\left(\hat{a}_1^\dagger\hat{a}_2\hat{a}_2^\dagger\hat{a}_1 + \hat{a}_2^\dagger\hat{a}_1\hat{a}_1^\dagger\hat{a}_2\right)$$

$$+T_{12}S_{11}^*S_{12}\left(\hat{a}_2^\dagger\hat{a}_2\hat{a}_1^\dagger\hat{a}_2 + \hat{a}_1^\dagger\hat{a}_2\hat{a}_2^\dagger\hat{a}_2 - \hat{a}_1^\dagger\hat{a}_1\hat{a}_1^\dagger\hat{a}_2 - \hat{a}_1^\dagger\hat{a}_2\hat{a}_1^\dagger\hat{a}_1\right)$$

$$+T_{12}S_{12}^*S_{11}\left(\hat{a}_2^\dagger\hat{a}_2\hat{a}_2^\dagger\hat{a}_1 + \hat{a}_2^\dagger\hat{a}_1\hat{a}_2^\dagger\hat{a}_2 - \hat{a}_1^\dagger\hat{a}_1\hat{a}_2^\dagger\hat{a}_1 - \hat{a}_2^\dagger\hat{a}_1\hat{a}_1^\dagger\hat{a}_1\right)$$

$$+(S_{11}^*S_{12})^2\hat{a}_1^\dagger\hat{a}_2\hat{a}_1^\dagger\hat{a}_2 + (S_{12}^*S_{11})^2\hat{a}_2^\dagger\hat{a}_1\hat{a}_2^\dagger\hat{a}_1 .$$

Calculating the mixed noise

Using the anticommutation relations, and calculating part by part:

$$\langle \hat{a}^\dagger_\alpha \hat{a}_\alpha \hat{a}^\dagger_\alpha \hat{a}_\alpha \rangle = \langle \hat{a}^\dagger_\alpha \left(1 - \hat{a}^\dagger_\alpha \hat{a}_\alpha\right) \hat{a}_\alpha \rangle = \langle \hat{a}^\dagger_\alpha \hat{a}_\alpha \rangle - \langle \hat{a}^\dagger_\alpha \hat{a}^\dagger_\alpha \hat{a}_\alpha \hat{a}_\alpha \rangle = f_\alpha - 0 = f_\alpha \, ,$$

$$\langle \hat{a}^\dagger_\alpha \hat{a}_\alpha \hat{a}^\dagger_\beta \hat{a}_\beta \rangle = \langle \hat{a}^\dagger_\alpha \hat{a}_\alpha \rangle \langle \hat{a}^\dagger_\beta \hat{a}_\beta \rangle = f_\alpha f_\beta, \quad \alpha \neq \beta \, ,$$

$$\langle \hat{a}^\dagger_\alpha \hat{a}_\beta \hat{a}^\dagger_\beta \hat{a}_\alpha \rangle = \langle \hat{a}^\dagger_\alpha \left(1 - \hat{a}^\dagger_\beta \hat{a}_\beta\right) \hat{a}_\alpha \rangle = \langle \hat{a}^\dagger_\alpha \hat{a}_\alpha \rangle - \langle \hat{a}^\dagger_\alpha \hat{a}^\dagger_\beta \hat{a}_\beta \hat{a}_\alpha \rangle =$$

$$= f_\alpha - \langle \hat{a}^\dagger_\alpha \hat{a}_\alpha \hat{a}^\dagger_\beta \hat{a}_\beta \rangle = f_\alpha - \langle \hat{a}^\dagger_\alpha \hat{a}_\alpha \rangle \langle \hat{a}^\dagger_\beta \hat{a}_\beta \rangle = f_\alpha (1 - f_\beta) \, , \; \alpha \neq \beta \, .$$

The last three lines on the square current operator give zero, because of different number of creation and annihilation operator with the same indices.

Calculating the mixed noise

This way we get the square current value:

$$\langle \hat{I}_1^2 \rangle / I_0^2 = T_{12}^2 (f_2 + f_1 - 2 f_1 f_2) + R_{11} T_{12} \{ f_1 (1 - f_2) + f_2 (1 - f_1) \}$$

And the mean square fluctuations:

$$\langle \delta I_1^2 \rangle / I_0^2 = \langle I_1^2 \rangle / I_0^2 - \langle I_1 \rangle^2 / I_0^2$$

$$= T_{12}^2 (f_2 + f_1 - 2 f_1 f_2) + R_{11} T_{12} \{ f_1 (1 - f_2) + f_2 (1 - f_1) \} - T_{12}^2 (f_2 - f_1)^2$$

$$= T_{12}^2 \{ f_1 (1 - f_1) + f_2 (1 - f_2) \} + R_{11} T_{12} \{ f_1 (1 - f_2) + f_2 (1 - f_1) \} .$$

Understanding the mixed noise

The first term: $$T_{12}^2\{f_1(1-f_1)+f_2(1-f_2)\}$$

- - This part originates from averaging those pairs of operators wich contribute to current.
- - Here the transmission reduces the electron flow and this way the noise, too.
- - This flcutuation is present only at non-zero temperature

$$\downarrow$$

We can consider this part as the thermal noise, but the dependence from the transmission probability is not the same as in the ballistic calculation.

$$\downarrow$$

We have to examine other parts of the mean square fluctuation, too.

Understanding the mixed noise

The next part is: $R_{11}T_{12}\{f_1(1-f_2)+f_2(1-f_1)\}$

- - This part vanishes at zero temperature
- - This part originates from averaging those pairs of operators wich do not contribute to current.
- - They correspond to two-particle processes.

We introduce the holes in the place of not occupied electrons.

- - An incoming electron/hole is reflected from the scatterer and another incoming hole/electron is transmitted.

As the fluctuations in the reservoirs, and the fluctuations caused by the scattering are independent, we can simply add them.

Understanding the mixed noise

Using these results one can rearrange our result:

$$\langle \delta I_1^2 \rangle / I_0^2 = T_{12}^2 \{ f_1(1 - f_1) + f_2(1 - f_2) \}$$

$$+ R_{11} T_{12} \{ f_1(1 - f_1 + f_1 - f_2) + f_2(1 - f_2 + f_2 - f_1) \} =$$

$$= \left(T_{12}^2 + R_{11} T_{12} \right) \{ f_1(1 - f_1) + f_2(1 - f_2) \}$$

Thermal noise

Shot noise

$$+ R_{11} T_{12} \{ f_1(f_1 - f_2) + f_2(f_2 - f_1) \} =$$

$$= T_{12} \{ f_1(1 - f_1) + f_2(1 - f_2) \} + R_{11} T_{12} (f_2 - f_1)^2 .$$

Current correlator

The connection beetwen different physical values is usually calculated in the form of correlation functions.

The current correlation function is defined this way:

$$P_{\alpha\beta}(t_1, t_2) = \frac{1}{2}\left\langle \Delta\hat{I}_\alpha(t_1)\Delta\hat{I}_\beta(t_2) + \Delta\hat{I}_\beta(t_2)\Delta\hat{I}_\alpha(t_1) \right\rangle$$

Where $$\Delta\hat{I}_\alpha = \hat{I}_\alpha - \langle \hat{I}_\alpha \rangle$$

Current correlator in frequency domain

To calculate the spectral contents we need to go to the frequency representation:

$$P_{\alpha\beta}(t_1, t_2) = \int_{-\infty}^{\infty} \frac{d\omega_1}{2\pi} e^{-i\omega_1 t_1} \int_{-\infty}^{\infty} \frac{d\omega_2}{2\pi} e^{-i\omega_2 t_2} P_{\alpha\beta}(\omega_1, \omega_2)$$

$$P_{\alpha\beta}(\omega_1, \omega_2) = \int_{-\infty}^{\infty} dt_1\, e^{i\omega_1 t_1} \int_{-\infty}^{\infty} dt_2\, e^{i\omega_2 t_2} P_{\alpha\beta}(t_1, t_2)$$

In stationary case instead of (t_1,t_2) the t_1-t_2 is enough

$$P_{\alpha\beta}(\omega_1, \omega_2) = 2\pi\, \delta(\omega_1 + \omega_2)\, \mathcal{P}_{\alpha\beta}(\omega_1)$$

The spectral noise power

The spectral noise power is defined as:

$$\mathcal{P}_{\alpha\beta}(\omega) = \int_{-\infty}^{\infty} dt\, e^{i\omega t}\, P_{\alpha\beta}(t) \qquad P_{\alpha\beta}(t) = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} e^{-i\omega t}\, \mathcal{P}_{\alpha\beta}(\omega)$$

If we measure the flcutuations in a frequency window:

$$\langle \delta I_\alpha^2 \rangle = \int_{-\Delta\omega/2}^{\Delta\omega/2} \frac{d\omega}{2\pi} \mathcal{P}_{\alpha\alpha}(\omega) \longrightarrow \frac{\langle \delta I_\alpha^2 \rangle}{\Delta\nu} = \mathcal{P}_{\alpha\alpha}(0)$$

If the scattering depends only a little from energy

Summary

- Definitions:
  - - Mean square fluctuations
  - - Thermal / Shot noise

- Detailed discussion:
  - - Thremal / Shot noise
  - - Mixed noise

- Introduction to correlation functions
  - - Correlation functions in time / freqency space
  - - Low energy-dependence case

Thank you for your attention!