Current noise in the case of continuous electron energy spectrum

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Recapitulation

- **Earlier**: mono-energetic carriers
  - Thermal noise due to equilibrium fluctuations of occupation numbers in reservoirs:
    $\langle \delta I_1^2 \rangle^{(Th)} / I_0^2 = T_{12}\{f_2(1 - f_2) + f_1(1 - f_1)\}$
  - Shot noise due to QM scattering processes:
    $\langle \delta I_1^2 \rangle^{(Sh)} / I_0^2 = R_{11} T_{12}(f_2 - f_1)^2$
- **Now**: Electrons with continuous E spectrum, one-dimensional leads connecting N_r reservoirs to mesoscopic sample

The current correlator

Current auto- ($\alpha = \beta$) and cross- ($\alpha \neq \beta$) correlators

$$P_{\alpha\beta}(t_1, t_2) = \frac{1}{2}\left\langle \Delta\hat{I}_\alpha(t_1)\Delta\hat{I}_\beta(t_2) + \Delta\hat{I}_\beta(t_2)\Delta\hat{I}_\alpha(t_1)\right\rangle$$

$$\Delta\hat{I}_\alpha = \hat{I}_\alpha - \langle\hat{I}_\alpha\rangle$$

- if $t_1 = t_2$ and $\alpha = \beta$, then $P_{\alpha\alpha}(t_1, t_1) = \langle\Delta\hat{I}_\alpha^2\rangle$ is the mean square fluctuation of current in lead α
- **Divergent** in systems with continuous unbounded spectrum $\Rightarrow$ Fourier-transform, measurement with $\Delta\omega$ window

The current correlator

In frequency domain

$$P_{\alpha\beta}(\omega_1, \omega_2) = \int_{-\infty}^{\infty} dt_1 e^{i\omega_1 t_1} \int_{-\infty}^{\infty} dt_2 e^{i\omega_2 t_2} P_{\alpha\beta}(t_1, t_2)$$

$$P_{\alpha\beta}(t_1, t_2) = \int_{-\infty}^{\infty} \frac{dt_1}{2\pi} e^{-i\omega_1 t_1} \int_{-\infty}^{\infty} \frac{dt_2}{2\pi} e^{-i\omega_2 t_2} P_{\alpha\beta}(\omega_1, \omega_2)$$

For stationary systems

- $P_{\alpha\beta}(t_1, t_2) = P_{\alpha\beta}(t_1 - t_2) = P_{\alpha\beta}(t) \Rightarrow$
- $P_{\alpha\beta}(\omega_1, \omega_2) = 2\pi\delta(\omega_1 + \omega_2)\mathcal{P}_{\alpha\beta}(\omega_1)$ (to be proven later!)
- $\mathcal{P}_{\alpha\beta}(\omega) = \int_{-\infty}^{\infty} dt e^{i\omega t} P_{\alpha\beta}(t)$ is the spectral noise power
- Measured current fluctuation $\langle \delta \hat{I}_\alpha \rangle = \int_{-\Delta\omega/2}^{\Delta\omega/2} d\omega \mathcal{P}_{\alpha\alpha}(\omega)$

The current correlator

Weak dependence on ω of SNP

- $\mathcal{P}_{\alpha\beta}(\omega) \approx \mathcal{P}_{\alpha\beta}(0)$
- $\frac{\langle \delta\hat{I}_\alpha \delta\hat{I}_\beta \rangle}{\Delta\nu} = \mathcal{P}_{\alpha\beta}(0)$ and $\frac{\langle \delta\hat{I}^2_\alpha \rangle}{\Delta\nu} = \mathcal{P}_{\alpha\alpha}(0)$, where $\Delta\nu = \frac{\Delta\omega}{2\pi}$
- To be compared with earlier results for thermal and shot noise of current (with two leads): $\frac{\langle \delta\hat{I}^2 \rangle^{(Th)}}{\Delta\nu} = k_B(T_1 + T_2)G$ and $\frac{\langle \delta\hat{I}^2 \rangle^{(Sh)}}{\Delta\nu} = |eVG|(1 - T_{12})$

Structure of the current correlator in the frequency domain

Correlator

$$P_{\alpha\beta}(\omega_1, \omega_2) = \frac{1}{2}\left\langle \Delta\hat{I}_\alpha(\omega_1)\Delta\hat{I}_\beta(\omega_2) + \Delta\hat{I}_\beta(\omega_2)\Delta\hat{I}_\alpha(\omega_1)\right\rangle$$

$$\text{where } \Delta\hat{I}_\alpha(\omega) = \hat{I}_\alpha(\omega) - \langle\hat{I}_\alpha(\omega)\rangle$$

We've learnt earlier, that ...

$$\hat{I}_\alpha(t) = \frac{e}{h}\iint dEdE' e^{\frac{E-E'}{\hbar}t}\{\hat{b}^\dagger_\alpha(E)\hat{b}_\alpha(E') - \hat{a}^\dagger_\alpha(E)\hat{a}_\alpha(E')\}$$

so...

$$\hat{I}_\alpha(\omega) = e\int_0^\infty dE\{\hat{b}^\dagger_\alpha(E)\hat{b}_\alpha(E+\hbar\omega) - \hat{a}^\dagger_\alpha(E)\hat{a}_\alpha(E+\hbar\omega)\}$$

Structure of the current correlator in the frequency domain

We separate $\hat{I}_\alpha$ into a scattered $\hat{I}_\alpha^{(out)} = e \int_0^\infty dE \hat{b}_\alpha^\dagger(E)\hat{b}_\alpha(E + \hbar\omega)$ and an incident $\hat{I}_\alpha^{(in)} = -e \int_0^\infty dE \hat{a}_\alpha^\dagger(E)\hat{a}_\alpha(E + \hbar\omega)$ current part, then

$$P_{\alpha\beta}(\omega_1, \omega_2) = \sum_{i,j=in,out} P_{\alpha\beta}^{(i,j)}(\omega_1, \omega_2)$$

where

$$P_{\alpha\beta}^{(i,j)}(\omega_1, \omega_2) = \frac{1}{2} \left\langle \Delta\hat{I}_\alpha^{(i)}(\omega_1)\Delta\hat{I}_\beta^{(j)}(\omega_2) + \Delta\hat{I}_\beta^{(j)}(\omega_2)\Delta\hat{I}_\alpha^{(i)}(\omega_1) \right\rangle$$

Correlator for incoming currents

$$P^{(in,in)}_{\alpha\beta}(\omega_1,\omega_2) = e^2 \iint dE_1 dE_2 \frac{J^{(in,in)}_{\alpha\beta}(E_{1,2},\omega_{1,2}) + J^{(in,in)}_{\beta\alpha}(E_{2,1},\omega_{2,1})}{2}$$

where:

$$J^{(in,in)}_{\alpha\beta}(E_{1,2},\omega_{1,2}) =$$
$$\langle\{\hat{a}^\dagger_\alpha(E_1)\hat{a}_\alpha(E_1+\hbar\omega_1) - \langle\, \hat{a}^\dagger_\alpha(E_1)\hat{a}_\alpha(E_1+\hbar\omega_1)\rangle\}\times$$
$$\{\hat{a}^\dagger_\beta(E_2)\hat{a}_\beta(E_2+\hbar\omega_2) - \langle\, \hat{a}^\dagger_\beta(E_2)\hat{a}_\beta(E_2+\hbar\omega_2)\rangle\}\rangle$$

From which:

$$J^{(in,in)}_{\alpha\beta}(E_{1,2},\omega_{1,2}) = \left\langle \hat{a}^\dagger_\alpha(E_1)\hat{a}_\beta(E_2+\hbar\omega_2)\right\rangle \left\langle\, \hat{a}_\alpha(E_1+\hbar\omega_1)\hat{a}^\dagger_\beta(E_2)\right\rangle$$

Correlator for incoming currents

Assuming uncorrelated reservoirs

$$\left\langle \hat{a}^\dagger_\alpha(E_1)\hat{a}_\beta(E_2 + \hbar\omega_2)\right\rangle = \delta_{\alpha\beta}\delta(E_1 - E_2 - \hbar\omega_2)f_\alpha(E_1)$$

$$\left\langle \hat{a}_\alpha(E_1 + \hbar\omega_1)\hat{a}^\dagger_\beta(E_2)\right\rangle = \delta_{\alpha\beta}\delta(E_1 + \hbar\omega_1 - E_2)\{1 - f_\alpha(E_1 + \hbar\omega_1)\}$$

we get uncorrelated incident currents

$$P^{(in,in)}_{\alpha\beta}(\omega_1, \omega_2) = 2\pi\delta(\omega_1 + \omega_2)\mathcal{P}^{(in,in)}_{\alpha\beta}(\omega_1)$$

$$\mathcal{P}^{(in,in)}_{\alpha\beta}(\omega_1) = \delta_{\alpha\beta}\frac{e^2}{h}\int_0^\infty dE_1\, F_{\alpha\alpha}(E_1, E_1 + \hbar\omega_1)$$

$$F_{\alpha\beta}(E, E') = \frac{1}{2}\left\langle f_\alpha(E)[1 - f_\beta(E')] + f_\beta(E')[1 - f_\alpha(E)]\right\rangle$$

Correlator for incoming and outgoing currents

Basically the same calculation, only for outgoing operators, we use

$$\hat{b}^\dagger_\beta(E) = \sum_{\gamma=1}^{N_r} S^*_{\beta\gamma}(E)\hat{a}^\dagger_\gamma(E) \quad \text{and} \quad \hat{b}_\beta(E) = \sum_{\gamma=1}^{N_r} S_{\beta\gamma}(E)\hat{a}_\gamma(E)$$

This way, we get

$$\mathcal{P}^{(in,out)}_{\alpha\beta}(\omega_1) = -\frac{e^2}{h}\int_0^\infty dE_1 F_{\alpha\alpha}(E_1, E_1+\hbar\omega_1)S^*_{\beta\alpha}(E_1+\hbar\omega_1)S_{\beta\alpha}(E_1)$$

and

$$\mathcal{P}^{(out,in)}_{\alpha\beta}(\omega_1) = -\frac{e^2}{h}\int_0^\infty dE_1 F_{\beta\beta}(E_1, E_1+\hbar\omega_1)S^*_{\alpha\beta}(E_1)S_{\alpha\beta}(E_1+\hbar\omega_1)$$

Correlations due to scattering!

Correlator for outgoing currents

Now there are only outgoing operators $\Rightarrow$ Two times as many scattering matrix components

$$\mathcal{P}^{(out,out)}_{\alpha\beta}(\omega_1) = \frac{e^2}{h}\int_0^\infty dE_1 \sum_{\gamma=1}^{N_r}\sum_{\delta=1}^{N_r} F_{\gamma\delta}(E_1, E_1+\hbar\omega_1)\times$$
$$S^*_{\alpha\gamma}(E_1)S_{\beta\gamma}(E_1)S_{\alpha\delta}(E_1+\hbar\omega_1)S^*_{\beta\delta}(E_1+\hbar\omega_1)$$

- Non-local dependence on all scattering amplitudes (phase coherent system)
- Frequency dependence of noise determined by internal and external factors:
  - Internal: energy dependence of scattering amplitudes
  - External: chemical potential, temperature of reservoirs (via Fermi-functions)

Energy independent scattering approximation

- Suppose $eV_{\alpha\beta} = \mu_\alpha - \mu_\beta$ and $T_\alpha = T_0$ for $\forall\alpha$ reservoirs
- $|eV_{\alpha\beta}|, k_B T_0, \hbar\omega \ll \mu_0$ very small excitations compared to Fermi-energy
- Suppose also little energy dependence of $S_{\alpha\beta}$ around $\mu_0 \Rightarrow$ $S_{\alpha\beta}(E) \approx S_{\alpha\beta}(E + \hbar\omega) \approx S_{\alpha\beta}(\mu_0)$

Energy integration only for E-dependent $F_{\alpha\beta}$

$$\int_0^\infty dEF_{\alpha\beta}(E, E + \hbar\omega) = \frac{eV_{\alpha\beta} + \hbar\omega}{2} \mathrm{cth}\left(\frac{eV_{\alpha\beta} + \hbar\omega}{2k_B T_0}\right)$$

Spectral noise power expression simplifies. . .

Energy independent scattering - Two leads case

$$\mathcal{P}_{11}(\omega) = \frac{e^2}{h}\left\{\hbar\omega\,\mathrm{cth}\left(\frac{\hbar\omega}{2k_B T_0}\right) T_{12}^2 + R_{11} T_{12} \times \right.$$
$$\left.\left[\frac{eV+\hbar\omega}{2}\mathrm{cth}\left(\frac{eV+\hbar\omega}{2k_B T_0}\right) + \frac{eV-\hbar\omega}{2}\mathrm{cth}\left(\frac{eV-\hbar\omega}{2k_B T_0}\right)\right]\right\}$$

where $V = V_{12} = -V_{21}$, $T_{12} = |S_{12}(\mu_0)|^2$, $R_{11} = |S_{11}(\mu_0)|^2$ and $\mathcal{P}_{12} = \mathcal{P}_{21} = -\mathcal{P}_{22} = -\mathcal{P}_{11}$ defines all correlations, $G = (e^2/h) T_{12}$, $I = VG$

$$\mathcal{P}_{11}(\omega) = \begin{cases} 2k_B T_0 G, & k_B T_0 \gg |eV|, \hbar\omega, \text{Thermal noise} \\ |eI| R_{11}, & |eV| \gg k_B T_0, \hbar\omega, \text{Shot noise} \\ \frac{e^2}{2\pi}|\omega| T_{12}, & \hbar\omega \gg |eV|, k_B T_0, \text{Quantum noise} \end{cases}$$

Low frequency measurements, Noise power

- Measurement with low frequencies $\Rightarrow$ fluctuation due to $\mathcal{P}_{\alpha\alpha}(\omega = 0)$, the noise power
- Form of $F_{\alpha\beta}$ simplifies, unitarity of $\hat{S}$
- $\mathcal{P}_{\alpha\beta}(0) = \mathcal{P}^{(Th)}_{\alpha\beta}(0) + \mathcal{P}^{(Sh)}_{\alpha\beta}(0)$

$$\mathcal{P}^{(Th)}_{\alpha\beta} = \frac{e^2}{h} \int_0^\infty dE \Bigg\{ \delta_{\alpha\beta} \Bigg[F_{\alpha\alpha}(E,E) + \sum_{\gamma=1}^{N_r} F_{\gamma\gamma} \left| S_{\alpha\gamma}(E) \right|^2 \Bigg] - F_{\alpha\alpha}(E,E) \left| S_{\beta\alpha}(E) \right|^2 - F_{\beta\beta}(E,E) \left| S_{\alpha\beta}(E) \right|^2 \Bigg\}$$

- Vanishes at $T = 0$, thermal noise power

Low frequency measurements, Noise power

$$\mathcal{P}^{(Sh)}_{\alpha\beta} = \frac{e^2}{h} \int_0^\infty dE \sum_{\gamma=1}^{N_r} \sum_{\delta=1}^{N_r} \frac{[f_\gamma(E) - f_\delta(E)]^2}{2} \times$$

$$\times S^*_{\alpha\gamma}(E) S_{\beta\gamma}(E) S_{\alpha\delta}(E) S^*_{\beta\delta}(E)$$

- Vanishes at zero current through the system, Shot noise power
- Both depends on bias and temperature
- Thermal (equilibrium) noise power: Same dependence on $\hat{S}$ as G $\Leftrightarrow$ Fluctuation-dissipation theorem
- Shot (non-equilibrium) noise power: more complicated dependence on matrix elements $\Rightarrow$ measurement tells more about the system

Properties of the noise power

- Noise power conservation law

$$\sum_{\alpha=1}^{N_r} \mathcal{P}^{(i)}_{\alpha\beta}(0) = 0 = \sum_{\beta=1}^{N_r} \mathcal{P}^{(i)}_{\alpha\beta}(0)$$

- for both incoming and outgoing indices and for i = Thermal, Shot or Total
- Due to unitarity of $\hat{S}$, to particle conservation
- Correlators are not independent at zero frequency $\Rightarrow$ some measured, others can be calculated

Properties of the noise power

- Sign rule for noise power (True to *Th* and *Sh* separately also)

$$\mathcal{P}_{\alpha\alpha}(0) \geq 0$$

$$\mathcal{P}_{\alpha\beta}(0) \leq 0, \; \alpha \neq \beta$$

- Auto-correlator positiveness is evident, mean square of a real quantity
- Cross-correlator negative sign comes from indivisible electrons (one incident electron is scattered to only one lead)
- negativity is also due to Pauli-principle: electrons with the same energy (spinless state) can only pass in the scattering channels one-by-one

Example: Scatterer with $N_r = 2$ leads

$$\mathcal{P}_{11}^{(Th)} = \mathcal{P}_{22}^{(Th)} = -\mathcal{P}_{12}^{(Th)} = -\mathcal{P}_{12}^{(Th)} = \mathcal{P}^{(Th)}$$

$$\mathcal{P}_{11}^{(Sh)} = \mathcal{P}_{22}^{(Sh)} = -\mathcal{P}_{12}^{(Sh)} = -\mathcal{P}_{12}^{(Sh)} = \mathcal{P}^{(Sh)}$$

where

$$\mathcal{P}^{(Th)} = \frac{e^2 k_B}{h} \int_0^\infty dE \left(-T_1 \frac{\partial f_1(E)}{\partial E} - T_2 \frac{\partial f_2(E)}{\partial E} \right) T_{12}(E)$$

$$\mathcal{P}^{(Sh)} = \frac{e^2}{h} \int_0^\infty dE \left[f_1(E) - f_2(E) \right]^2 T_{12}(E) R_{11}(E)$$

Energy dependence of transmission coefficients determines strongly the V and T dependence of noise $\Rightarrow$ former results as special cases

Fano-factor, measure of correlation between carriers

- The Fano-factor: $F = \frac{\mathcal{P}^{(Sh)}}{|qI|}$
- For statistically independent carriers $F = 1$
- Simplest case: $T_{12} = const$, $|eV| \gg k_B T$, $\mathcal{P}^{(Sh)} = |eI| R_{11}(\mu_0)$ $\Rightarrow F = 1 - T_{12} < 1$
- If $T_{12} \to 0$, then $F \approx 1 \Rightarrow$ as G T_{12} in case of small conductance, uncorrelated carriers carry the current
- Correlations reduce Fano-factor due to Pauli-exclusion (passing one-by-one)

Thank you for your attention!