## Quantum mechanics 1.

Febr. 14.

1. problem (Maximilian Buchta)
   Determine the energy levels and wavefunction of a particle in an infinite potential well!

2. problem (Daniel Brunsch)
   Show that the derivative of the wavefunction remains continuous if the one dimensional potential has only a finite jump. What is the boundary condition for a one dimensional Dirac-delta potential? Using this, evaluate the bound state energy of a particle in an attractive Dirac-delta potential!

3. problem (Lennart Hucht, thank you Mohamad)
   Find the bound states of the potential $V(x) = -V_0$ for $|x| < a$ and zero otherwise, $V_0 > 0$, $a > 0$. Discuss the parity of the solutions as well as the case of an infinite potential well.

Febr. 21.

1. problem (Yossarian Nick Liebsch)
   Demonstrate that for a one dimensional potential, the solutions of the time independent Schrödinger equation are non-degenerate!

2. problem (Tobias Lojewski)
   Show that for an even potential (i.e. $V(x) = V(-x)$), the eigenstates can be classified according to their parity, namely they will be either even or odd!

3. problem (Niklas Nickig)
   Determine the stationary solutions of a particle moving in a harmonic potential using Sommerfeld's polynomial method!
   $$-\frac{\hbar^2}{2m}\frac{d^2}{dx^2}\varphi_n(x) + \frac{1}{2}m\omega^2 x^2 \varphi_n(x) = E_n \varphi_n(x)$$

4. problem (Nils Herrmann Voellings)
   Determine the reflection and transmission coefficients of a particle through a $V_0 > 0$ potential step for $E \leq V_0$ és $E > V_0$!

Mar. 7.

1. problem (Maximilian Buchta)
   Calculate the transmission and reflection coefficients for the potential ($a > 0$, $\gamma > 0$):
   $$V(x) = \begin{cases} \gamma\delta(x) & \text{ha} \quad x < a \\ \infty & \text{ha} \quad x \geq a \end{cases}$$

2. problem (Daniel Brunsch)
   Calculate the transmission and reflection coefficients for the $\gamma\delta(x)$ potential!

3. problem (Lennart Hucht)
   Prove the following identity:
   $$[AB, C] = A[B, C] + [A, C]B \ ,$$
   where $A$, $B$ and $C$ linear operators!

4. problem (Yossarian Nick Liebsch)
   Assume that two operators commute as
   $$[\hat{A}, \hat{B}] = 0$$
   Show that for the non-degenerate case, the eigenstates of $\hat{A}$ are also eigenstates of $\hat{B}$:
   $$\hat{A}|u\rangle = a|u\rangle \qquad \hat{B}|u\rangle = b|u\rangle \qquad a, b \in \mathcal{C}$$

5. problem (Tobias Lojewski)
Show that the momentum operator,

$$\hat{p} = \frac{\hbar}{i}\frac{d}{dx}$$

is hermitian! Demonstrate that the shift operator by distance $X$ can be written as

$$T(X) = e^{\frac{i}{\hbar}X\hat{p}}$$

Mar. 14.

1. problem (Niklas Nickig)
Consider the Hamilton operator of a linear harmonic oscillator:

$$H = \frac{p^2}{2m} + \frac{1}{2}m\omega^2 x^2$$

Introduce $A = \alpha x + \beta p$ and $A^+ = \alpha^* x + \beta^* p$ operators. Determine the $\alpha$, $\beta$ parameters such that the Hamiltonian is rewritten as

$$H = A^+ A + C \ ,$$

with constant $C$. Using the $A$, $A^+$ operators, introduce $a$, $a^+$ dimensionless operators as

$$a = \frac{1}{\sqrt{\hbar\omega}}A, \qquad a^+ = \frac{1}{\sqrt{\hbar\omega}}A^+$$

Show that the Hamiltonian can be recast as:

$$H = \hbar\omega\left(a^+ a + \frac{1}{2}\right) \ .$$

2. problem (Nils Herrmann Voellings)
Using the previous exercise, determine the commutation relations:

$$[a, a^+] =, \qquad [H, a] =, \qquad [H, a^+] =$$

Using these, calculate the energy spectrum and eigenstates of the harmonic oscillator.

3. problem (Maximilian Buchta)
Using the $a$, $a^+$ ladder operators, determine the following expectation values:

$$\langle n|x|n\rangle = \qquad \langle n|p|n\rangle = \qquad \langle n|x^2|n\rangle = \qquad \langle n|p^2|n\rangle =$$

Show that for a harmonic oscillator, the expectation value of the kinetic energy is the same as that of the potential energy in a given eigenstate (Virial theorem).

4. problem (Daniel Brunsch)
Determine the matrix representation of the position $x$ and momentum $p$ operator in the basis of the eigenfunctions of the harmonic oscillator!

Mar. 21.

1. problem (Lennart Hucht)
Using the ladder operators, determine the expectation value of $\langle n|x^4|n\rangle$!

2. problem (Yossarian Nick Liebsch)
(Virial theorem)
Determine the commutation relations

$$[p_i, V] \qquad [H, x_i] \qquad [H, p_i],$$

where $H = \frac{\mathbf{p}^2}{2m} + V$ is the Hamilton operator! What is the quantum mechanical time derivative of the momentum?

3. problem (Tobias Lojewski)
   Demonstrate that for an arbitrary operator $A$,

$$\langle n|[H,A]|n\rangle = 0, \qquad \text{if} \qquad H|n\rangle = \varepsilon_n|n\rangle.$$

   Using this as the $\langle n|[H,\mathbf{rp}]|n\rangle = 0$ identity, prove the virial theorem:

$$\langle n|\frac{\mathbf{p}^2}{2m}|n\rangle = \langle n|\mathbf{r}\nabla V|n\rangle \text{ !}$$

   What does it predict for a linear harmonic oscillator?

4. problem (Niklas Nickig)
   Determine the ground state wavefunction of the harmonic oscillator using the lowering operator on both the coordinate and momentum representation!

$$a|0\rangle = 0 \qquad a = \frac{1}{\sqrt{2}}\left(\frac{x}{x_0} + i\frac{p}{p_0}\right) \qquad x_0 = \sqrt{\frac{\hbar}{m\omega}} \qquad p_0 = \sqrt{\hbar m\omega}$$

Mar. 28.

1. problem (Nils Herrmann Voellings)
   Prove the $\Delta x \Delta p \geq \hbar/2$ uncertainty principle for a harmonic oscillator using explicit calculations. Based on this principle, show that the minimal energy of the oscillator cannot be smaller than $\hbar\omega/2$!

2. problem (Maximilian Buchta)
   Define the $|\alpha\rangle = \sum_{n=0}^{\infty} c_n|n\rangle$ *coherent* state such that is satisfies the eigenvalue equation $a|\alpha\rangle = \alpha|\alpha\rangle$ with $\alpha$ a complex number, and $|n\rangle$ is an eigenstate of a harmonic oscillator, $a$ is the lowering operator.
   Determine the $c_n$ coefficients! Write the normalized $|\alpha\rangle$ wavefunction as $\exp(O)|0\rangle$, and calculate the operator $O$!

3. problem (Daniel Brunsch)
   Show that the time evolution of the coherent state, $\exp(iHt/\hbar)|\alpha\rangle$ remains an eigenstate of $a$ and determine its eigenvalue!

4. problem (Lennart Hucht)
   Determine the expectation value of the energy, $\langle\alpha|H|\alpha\rangle$ for a harmonic oscillator!

5. problem (Yossarian Nick Liebsch)
   Evaluate the uncertainty principle using a coherent state!

April 18.

1. problem (Tobias Lojewski)
   Demonstrate using the Hausdorff expansion, that by rotating $L_z$ around $x$ yields $L_y$!

$$L_y = e^{\frac{i}{\hbar}\frac{\pi}{2}L_x} L_z e^{-\frac{i}{\hbar}\frac{\pi}{2}L_x}$$

2. problem (Niklas Nickig)
   Using the common eigenstates of $\mathbf{L}^2$ and $L_z$, denoted by $|l,m\rangle$, calculate the variance of $L_x$ and $L_y$!

3. problem (Nils Herrmann Voellings)
   Construct the matrix representation of the operators $L_x$ and $L_y$ using the common eigenstates of $L^2$ and $L_z$ within the

$$\{|\,1,-1\rangle, |\,1,0\rangle, |\,1,1\rangle\}$$

   three dimensional subspace. What are the eigenvalues?

Show that
$$L_\pm|l,m\rangle = \hbar\sqrt{l(l+1)-m(m\pm 1)}|l,m\pm 1\rangle$$
using
$$L_+L_- = \mathbf{L}^2 - L_z^2 + \hbar L_z \qquad\qquad L_-L_+ = \mathbf{L}^2 - L_z^2 - \hbar L_z$$

4. problem (Maximilian Buchta)
Let us consider the isotropic three dimensional harmonic oscillator:
$$H = \frac{p^2}{2m} + \frac{1}{2}m\omega^2\mathbf{r}^2\ .$$
Upon introducing the ladder operators,
$$a_i = \frac{1}{\sqrt{2}}\left(\frac{r_i}{x_0} + i\frac{p_i}{p_0}\right) \qquad a_i^+ = \frac{1}{\sqrt{2}}\left(\frac{r_i}{x_0} - i\frac{p_i}{p_0}\right) \qquad x_0 = \sqrt{\frac{\hbar}{m\omega}},\ p_0 = \sqrt{\hbar m\omega}$$
show that
$$\mathbf{L} = \frac{\hbar}{i}\mathbf{a}^+ \times \mathbf{a}\,!$$

April 25.

1. problem (Daniel Brunsch)
Show that the radial momentum operator,
$$p_r = \frac{\hbar}{i}\left(\frac{\partial}{\partial r} + \frac{1}{r}\right)$$
is Hermitian!

2. problem (Lennart Hucht)
Determine the splitting of the $n$=2 level of the hydrogen atom in a homogeneous electric field in the $z$ direction, to first order in perturbation theory! (First order Stark effect)
*Hint*:
$$\psi_{200}(\vec{r}) = \frac{2}{(2a_0)^{3/2}}\left(1-\frac{r}{2a_0}\right)\exp\left(-\frac{r}{2a_0}\right)Y_0^0(\vartheta,\varphi)\ ,$$
$$\psi_{21m}(\vec{r}) = \frac{2}{\sqrt{3}(2a_0)^{3/2}}\frac{r}{2a_0}\exp\left(-\frac{r}{2a_0}\right)Y_1^m(\vartheta,\varphi)\ ,$$
$$Y_0^0(\vartheta,\varphi) = \frac{1}{\sqrt{4\pi}}\ , Y_1^0(\vartheta,\varphi) = \sqrt{\frac{3}{4\pi}}\cos(\vartheta)\ ,$$
$$Y_1^{-1}(\vartheta,\varphi) = \sqrt{\frac{3}{8\pi}}\sin(\vartheta)e^{-i\varphi}\ , Y_1^1(\vartheta,\varphi) = -\sqrt{\frac{3}{8\pi}}\sin(\vartheta)e^{i\varphi}\ ,$$
$$\int_0^\infty x^n e^{-\alpha x}dx = \frac{n!}{\alpha^{n+1}}\quad .$$

3. problem (Yossarian Nick Liebsch)
Use the virial theorem for the hydrogen atom ( $\langle nlm|[H, rp_r]|nlm\rangle = 0$) to calculate the expectation value of $\langle 1/r\rangle$!

4. problem (Tobias Lojewski)
Using the commutation relations and $\langle nlm|[H, r^2p_r]|nlm\rangle = 0$, determine the expectation value of $\langle nlm|r|nlm\rangle$!

May 2.

1. problem (Niklas Nickig)
Using the variational principle, determine the ground state wavefunction and energy of the hydrogen atom using $H = \frac{p_r^2}{2m} + \frac{L^2}{2mr^2} - \frac{\alpha}{r}$! The variational wavefunction is $\Psi(r) = N\exp(-\beta r)$.

2. problem (Nils Herrmann Voellings)
Consider an infinite potential well as
$$V(x) = \begin{cases} 0 & \text{for} \quad |x| < a \\ \infty & \text{for} \quad |x| \geq a \end{cases}$$
Using the (unnormalized) variational wavefunction $\Psi_\beta(x) = a^\beta - |x|^\beta$, determine the optimal $\beta$ and the corresponding ground state energy.

May 9.

1. problem (Maximilian Buchta)
   Consider the Pauli matrices $\sigma_{x,y,z}$. Using $\sigma_i\sigma_j = \delta_{ij} + i\epsilon_{ijk}\sigma_k$, derive $(\boldsymbol{\sigma}\mathbf{a})^2 = \mathbf{a}^2$! Calculate the eigenvectors of the matrix $\mathbf{n}\boldsymbol{\sigma}$, where $\mathbf{n} \in \mathcal{R}^3$ and $\mathbf{n}^2 = 1$!

2. problem (Daniel Brunsch)
   Consider a two dimensional harmonic oscillator with
   $$H = \frac{p_x^2}{2m} + \frac{p_y^2}{2m} + \frac{1}{2}m\omega^2(x^2 + y^2).$$
   Calculate the energy of the first excited state within first order perturbation theory in the presence of the potential $W = m\omega^2\alpha xy$. Use $x = x_0(a + a^+)/\sqrt{2}$, where $x_0 = \sqrt{\hbar/m\omega}$.

3. problem (Lennart Hucht)
   Consider a hydrogen atom in its ground state, placed in a homogeneous external magnetic field in the $z$ direction. Due to the external magnetic field, the spin-degenerate wavefunction splits and the spinor wavefunction assumes the form
   $$\psi = \begin{pmatrix} 0 \\ \varphi_0 \end{pmatrix} ,$$
   where $\varphi_0$ is the ground state wavefunction in the absence of the field. The Hamiltonian of the system reads as
   $$H = \frac{\mathbf{p}^2}{2m} - \frac{ke^2}{r} + \frac{\mu_B}{\hbar}\left(L_z + 2S_z\right) B \, .$$
   What is the probability to find the system in the $S_z$ eigenstate with eigenenergy $+\hbar/2$ due to an additional, weak external magnetic field in the $x$ direction ($\alpha > 0$):
   $$B_x = \begin{cases} B_0 e^{\alpha t}\sin(\omega t) & \text{ha} \quad t < 0 \\ B_0 \sin(\omega t) & \text{ha} \quad t \geqslant 0 \end{cases} .$$

4. problem (Yossarian Nick Liebsch )
   A linear harmonic oscillator is placed within the sheets of a discharging capacitor. The perturbing potential takes the form:
   $$V(t) = \mathcal{E} q x e^{-\alpha t} \, .$$
   What is the probability to find the system, which was initially in its ground state initially, in the $n$th excited state within first order time dependent perturbation theory?